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OF THIN ELASTIC FLAT PLATES
AND SHELLS

by K. A. V. Pandalia



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OF THIN ELASTIC FLAT PLATES
AND SHELLS

By

K.A.V. Pandalai*

School of Engineering and Applied Sciences
The George Washington University
Washington, D.C.

* Senior Foreign Scientist Fellow of
The National Science Foundation and
Visiting Research Professor at the
George Washington University. On
leave of absence from the Indian
Institute of Technology, Madras, India.

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ABSTRACT

The general equations governing the large amplitude flexural vibration of any thin elastic shell using curvilinear orthogonal coordinates are derived in this report. These equations consist of two coupled nonlinear partial differential equations in the normal displacement w and the stress function F . From these equations, the governing equations for the case of shells of revolution or flat plates can be readily obtained as special cases. The material of the shell or plate is isotropic and homogeneous and Hooke's law for the two dimensional case is valid. Modal equations are derived on the basis of a series for the assumed vibration mode. A typical term of the series is taken to be separable in the space coordinates and time. The nature of the modal equation obtained for the case of flat plates explains the reason why the nonlinearity associated with the large amplitude flexural vibration of thin flat plates of any geometry is of the hardening type. Similarly, the modal equation applicable for the case of any thin shell shows the reasons why the softening type of nonlinearity is quite possible in the case of thin elastic shells. The difference between the hardening type of nonlinearity in the case of flat plates and straight beams and the softening type of nonlinearity in the case of shells and rings can in general be traced to the curvature (or lack of it) of the undeformed median surface of the structure concerned.

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Introduction.

The nonlinear (large amplitude) flexural vibration of plates and shells have received considerable attention in the last 20 years. This area is of great importance and interest in the case of structures of low flexural rigidity which as a consequence are easily deformable. The importance of this investigation arises from the fact that when the flexural vibrations involve large amplitudes, the frequency of free or forced vibration is very much dependant upon the amplitude. Significant changes in the frequency corresponding to the small-amplitude vibration can occur when the amplitudes involved are no longer small so that the resulting governing equations of the problem are nonlinear.

A somewhat comprehensive bibliography of recent papers published in the field of large amplitude flexural vibration of thin elastic flat plates and shells is given at the end of this report. This bibliography which gives 82 references on plates and 57 references on shells is taken from a recent survey paper (to be published) titled 'Nonlinear Flexural Vibrations of Certain Deformable Bodies' co-authored by this author (Ref.6).

As can be seen from the bibliography mentioned earlier, the existing literature as far as the nonlinear

flexural vibration of thin flat plates is concerned, relate to plates of rectangular, circular, elliptical, triangular and parallelogrammic geometry. Most of the investigations carried out so far deal with isotropic plates. In order to obtain the solutions of problems of plate geometries other than those already investigated, the governing equations for the large amplitude flexural vibration of thin elastic flat plates in orthogonal curvilinear co-ordinates are necessary. Therefore these equations have been derived in the next section where the necessary modification to tackle the problem of buckling of thin elastic flat plates is also indicated.

In the area of shells, the existing literature on the large amplitude flexural vibrations relate to circular cylindrical shells, spherical and conical shells. Recently under the guidance of the author, work was carried out on the nonlinear flexural vibrations of orthotropic oval cylindrical shells. It is of interest to study the behaviour of thin elastic shells of geometry other than those mentioned above. To this end, the general equations governing the nonlinear flexural vibration of thin shells of revolution have been obtained in Section 3. It is proposed to use these equations in the case of the ogival shell, torus shell and ellipsoidal shell of revolution. The governing equations mentioned above are suitably modified to tackle the problem of buckling of thin elastic shells of revolution.

Having derived the governing equations for the nonlinear flexural vibration of shells of revolution as well as of thin elastic flat plates using curvilinear orthogonal coordinates, the governing equations for the nonlinear flexural vibration of ANY thin elastic shell can be readily obtained. From these very general equations, the governing equations applicable for shells of revolution or thin flat plates can be obtained as special cases.

In the last but one section, on the basis of an assumed vibration mode of the product type (product of function of time and function of space variables), the modal equation is determined for the general case of thin elastic shells. The modal equation is then specialised for the case of flat plates and beams. The characteristic features of the modal equation as applicable to thin shells on the one hand and thin flat plates and straight beams on the other hand are discussed. It is conclusively shown that in the case of flat plates and beams, the nature of the modal equation is such that the nonlinearity of the hardening type results.

In the last section, the governing equations for the postbuckling behavior of flat plates and thin shells obtained as a byproduct of the study of the problem of large amplitude flexural vibration are examined. It is shown that in general

the set of algebraic equations in the coefficients of the assumed series for the postbuckling mode are a set of coupled cubic equations in these coefficients in the case of both plates and shells. However, in the absence of any lateral load acting on the flat plate, the set of algebraic equations in the coefficients reduces to quadratic equations.

Section 2.

In this section, the governing equations for the large amplitude flexural vibration of thin elastic flat plates in curvilinear orthogonal co-ordinates α_1 and α_2 are derived. α_1 , α_2 and η form as usual a right-handed orthogonal coordinate system.

An elemental length ds on the flat plate median surface is given by

$$(ds)^2 = (A_1 d\alpha_1)^2 + (A_2 d\alpha_2)^2 \quad (2.1)$$

where, A_1 and A_2 are Lamé parameters of the median surface and are functions of the curvilinear orthogonal coordinates α_1 and α_2 . For a point off the median surface at a distance η away, the extensional and shear strains are given by [1] , [2]

$$\begin{aligned} \epsilon_{11} &= \epsilon_1 + \eta \kappa_{11} \\ \epsilon_{22} &= \epsilon_2 + \eta \kappa_{22} \\ \gamma_{12} &= \omega + 2\eta \kappa_{12} \end{aligned} \quad (2.2)$$

where, the median surface strains ϵ_1, ϵ_2 and ω are given by

$$\begin{aligned}
 \epsilon_1 &= \left\{ \left[\frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) v \right] + \frac{1}{2} \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right)^2 \right\} \\
 \epsilon_2 &= \left\{ \left[\frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{1}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) u \right] + \frac{1}{2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right)^2 \right\} \\
 \omega &= \left\{ \left(\frac{A_2}{A_1} \right) \frac{\partial (v/A_2)}{\partial \alpha_1} + \left(\frac{A_1}{A_2} \right) \frac{\partial (u/A_1)}{\partial \alpha_2} + \frac{1}{A_1 A_2} \left(\frac{\partial w}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \right\}
 \end{aligned} \tag{2.3}$$

The curvature expressions κ_{11} , κ_{22} and κ_{12} are given by:

$$\begin{aligned}
 \kappa_{11} &= -\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) - \frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) \\
 \kappa_{22} &= -\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) - \frac{1}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) \\
 \kappa_{12} &= -\frac{1}{A_1 A_2} \left[\frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} - \frac{1}{A_1} \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{\partial w}{\partial \alpha_1} \right) - \frac{1}{A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \right]
 \end{aligned} \tag{2.4}$$

u , v and w are as usual the components of the displacement vector in the orthogonal co-ordinate directions α_1 , α_2 and η respectively. From equations (2.4) it follows readily that,

$$(\kappa_{11} + \kappa_{22}) = -\nabla^2 w \equiv -\frac{1}{A_1 A_2} \left\{ \frac{\partial}{\partial \alpha_1} \left[\left(\frac{A_2}{A_1} \right) \frac{\partial w}{\partial \alpha_1} \right] + \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1}{A_2} \right) \frac{\partial w}{\partial \alpha_2} \right] \right\} \tag{2.5}$$

For the isotropic case, the strain energy u can be written as (See Equation (9.12) of Ref.2):

$$u = u_1 + u_2 \quad (2.6)$$

where,

$u_1 =$ Extensional strain energy

$$= \frac{C}{2} \int_A [(\epsilon_1 + \epsilon_2)^2 - 2(1-\nu)(\epsilon_1\epsilon_2 - \frac{\omega^2}{4})] dA \quad (2.7)$$

and

$u_2 =$ Bending strain energy

$$= \frac{D}{2} \int_A \{(\kappa_{11} + \kappa_{22})^2 - 2(1-\nu)[\kappa_{11}\kappa_{22} - (\kappa_{12})^2]\} dA \quad (2.8)$$

In the above, $C = \frac{Eh}{(1-\nu^2)}$ and $D = \frac{Eh^3}{12(1-\nu^2)}$.

In view of Equation (2.5), Equation (2.8) can be written as

$$u_2 = \frac{D}{2} \int_A \{(\nabla^2 w)^2 - 2(1-\nu)[\kappa_{11}\kappa_{22} - (\kappa_{12})^2]\} dA \quad (2.9)$$

In the above integrals, the elemental area dA of the median surface of the flat plate is given by

$$dA = A_1 A_2 d\alpha_1 d\alpha_2 \quad (2.10)$$

From the principle of minimum total potential, $\delta(u-W)=0$ where the work done by the applied loads is given by

$$W = \int_A (\Phi u + \Theta v + Z w) dA \quad (2.11)$$

In the above, Φ , Θ and Z are the components of the applied load in the coordinate directions α_1 , α_2 and β respectively. The three equations of equilibrium obtained from the principle of minimum total potential can be shown to be:

$$\frac{\partial}{\partial \alpha_1} (A_2 N_{11}) - N_{22} \left(\frac{\partial A_2}{\partial \alpha_1} \right) + \frac{1}{A_1} \frac{\partial}{\partial \alpha_2} (A_1^2 N_{12}) + A_1 A_2 \Phi = 0 \quad (2.12)$$

$$\frac{\partial}{\partial \alpha_2} (A_1 N_{22}) - N_{11} \left(\frac{\partial A_1}{\partial \alpha_2} \right) + \frac{1}{A_2} \frac{\partial}{\partial \alpha_1} (A_2^2 N_{12}) + A_1 A_2 \Theta = 0 \quad (2.13)$$

$$\begin{aligned} D[(A_1 A_2) \nabla^4 w - (1-\nu) L] - \left\{ \frac{\partial}{\partial \alpha_1} \left[\left(\frac{A_2}{A_1} \right) N_{11} \frac{\partial w}{\partial \alpha_1} \right] + \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1}{A_2} \right) N_{22} \frac{\partial w}{\partial \alpha_2} \right] \right. \\ \left. + 2 N_{12} \frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} + \left(\frac{\partial N_{12}}{\partial \alpha_1} \right) \frac{\partial w}{\partial \alpha_2} + \left(\frac{\partial N_{12}}{\partial \alpha_2} \right) \frac{\partial w}{\partial \alpha_1} \right\} - A_1 A_2 Z = 0 \end{aligned} \quad (2.14)$$

In the above, $\nabla^4 w = \nabla^2(\nabla^2 w)$ and N_{11} , N_{22} and N_{12} are the stress resultants defined by $N_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} d\beta$.

Also, the function $L(A_1, A_2, w)$ stands for

$$L = \left\{ -\frac{\partial}{\partial \alpha_1} \left[\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} (A_2 \kappa_{22}) \right] - \frac{\partial}{\partial \alpha_2} \left[\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} (A_1 \kappa_{11}) \right] + 2 \frac{\partial^2 \kappa_{12}}{\partial \alpha_1 \partial \alpha_2} \right\} \\ + \left\{ \frac{\partial}{\partial \alpha_1} \left[\left(\frac{\kappa_{11}}{A_1} \right) \frac{\partial A_2}{\partial \alpha_1} \right] + \frac{\partial}{\partial \alpha_2} \left[\left(\frac{\kappa_{22}}{A_2} \right) \frac{\partial A_1}{\partial \alpha_2} \right] + \frac{\partial}{\partial \alpha_2} \left[\left(\frac{\kappa_{12}}{A_2} \right) \frac{\partial A_2}{\partial \alpha_1} \right] \right\}$$

(2.15)

where, the curvatures κ_{11} , κ_{22} and κ_{12} are defined by Equations (2.4) in terms of the displacement component w .

It is of importance to recognise that Equations (2.12) and (2.13) are the equations of inplane equilibrium and are exactly the same as those developed by Mushtari in the well-known Mushtari-Vlasov approximation of thin shell theory (See Section 17, p.84 of Ref.(2)).

The expression inside the brace $\{ \dots \}$ appearing in Equation (2.14) can be simplified by means of Equations (2.12) and (2.13) wherein the load components $\bar{\Phi}$ and \bar{H} are neglected. Thus Equation (2.14) can be written as:

$$D \nabla^4 W = \frac{D(1-\nu)}{A_1 A_2} L - N_{11} \left[\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) + \left(\frac{1}{A_1 A_2} \right) \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) \right] \\ - N_{22} \left[\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) + \left(\frac{1}{A_1 A_2} \right) \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) \right] \\ - 2 N_{12} \left[\frac{1}{A_1 A_2} \frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} - \left(\frac{1}{A_1 A_2} \right) \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) - \frac{1}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) \right] = Z$$

(2.16)

In view of Equations (2.4), the above equation can also be written as

$$DV^4W - \frac{D(1-\nu)}{A_1 A_2} L + N_{11} \mathcal{K}_{11} + N_{22} \mathcal{K}_{22} + 2N_{12} \mathcal{K}_{12} = Z \quad (2.17)$$

The function L given by Equations (2.15) if written out purely in terms of W will be somewhat complicated and lengthy. However, under certain conditions - to be discussed below - this function L can be shown to be zero. For example, if the Lamé parameters A_1 and A_2 are such that the products of the derivatives of A_1 (or A_2) with

$$\frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} \left(\text{or } \frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} \right) \quad \text{or their derivatives can be}$$

neglected in comparison with the terms of the type

$$\frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} \right) \quad \text{or} \quad \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} \right)$$

then, the function L defined by Equation (2.15) can be shown to vanish. In such a case, the curvature expressions given by Equation (2.4) also must be simplified to read:

$$\begin{aligned} \mathcal{K}_{11} &\approx -\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} \right) ; & \mathcal{K}_{22} &\approx -\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} \right); \\ \mathcal{K}_{12} &\approx -\frac{1}{A_1 A_2} \frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_2} \end{aligned}$$

Consequently Equation (2.17) simplifies to

$$\begin{aligned} D\nabla^4 W - N_{11} \left[\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} \right) \right] - N_{22} \left[\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} \right) \right] \\ - 2N_{12} \left[\frac{1}{A_1 A_2} \frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_2} \right] = Z \end{aligned} \quad (2.19)$$

Another way to look at this approximation (that is, $L = 0$) is to consider the expression for the strain energy U as given by Equation (2.6) from which it is seen that U consists of two parts, one proportional to the extensional

rigidity $C = \frac{Eh}{(1-\nu^2)}$ and the second part proportional to the bending rigidity $D = \frac{Eh^3}{12(1-\nu^2)}$. From Equation

(2.9) it is clear that the bending strain energy U_2

depends upon two quantities, namely, $(\nabla^2 W)^2$ and

$2(1-\nu) [N_{11} N_{22} - (N_{12})^2]$. If the integral of the

term $2(1-\nu) [N_{11} N_{22} - (N_{12})^2]$ is neglected in

comparison to the integral of $(\nabla^2 W)^2$, then $U_2 \approx \frac{D}{2} \int_A (\nabla^2 W)^2 dA$

and in such a case also, L will vanish.

The stress resultants N_{11} , N_{22} and N_{12} are functions of the stress function $F(\alpha_1, \alpha_2)$ as defined later on so that Equation (2.19) represents an equation involving

W and F. If the terms involving coupling between W and F are neglected, Equation (2.19) reduces to the well-known equation of the linear bending theory of isotropic flat plates, namely, $D\nabla^4 W = Z$.

It has already been mentioned that Equations (2.12) and (2.13) are the equations of inplane equilibrium and that they are the same as those developed by Mushtari in the Mushtari-Vlasov approximation of thin shell theory [2].

The stress function $F(\alpha_1, \alpha_2)$ is so defined as to satisfy Equations (2.12) and (2.13) exactly and the relationship between the stress resultants (N_{11} , N_{22} and N_{12}) and the stress function F are exactly the same as those defined by Vlasov in the Mushtari-Vlasov approximation of thin shell theory (See Equation (17.8) of Ref.(2)). Thus,

$$\begin{aligned} N_{11} &= -\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial F}{\partial \alpha_2} \right) - \frac{1}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_1} \frac{\partial F}{\partial \alpha_1} \right) \\ N_{22} &= -\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial F}{\partial \alpha_1} \right) - \frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_2} \frac{\partial F}{\partial \alpha_2} \right) \\ N_{12} &= \frac{1}{A_1 A_2} \left[\frac{\partial^2 F}{\partial \alpha_1 \partial \alpha_2} - \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_1} \frac{\partial F}{\partial \alpha_1} \right) - \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_2} \frac{\partial F}{\partial \alpha_2} \right) \right] \end{aligned} \quad (2.20)$$

obviously,

$$(N_{11} + N_{22}) = -\nabla^2 F \equiv -\frac{1}{A_1 A_2} \left\{ \frac{\partial}{\partial \alpha_1} \left[\left(\frac{A_2}{A_1} \right) \frac{\partial F}{\partial \alpha_1} \right] + \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1}{A_2} \right) \frac{\partial F}{\partial \alpha_2} \right] \right\} \quad (2.21)$$

Because of Equations (2.20), Equation (2.16) or (2.19) represents one equation involving coupling between W and F . Before we proceed to obtain the second equation involving W and F , it is worth noting that Equation (2.19) represents the governing equation of plate buckling provided the signs in front of N_{11} , N_{22} and N_{12} are changed from minus to plus in Equation (2.19). This is obviously because for buckling problems, N_{11} and N_{22} will be compressive (and similarly for N_{12}) so that for plate buckling, the governing equation becomes:

$$D\nabla^4 W + N_{11} \left[\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} \right) \right] + N_{22} \left[\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} \right) \right] + 2N_{12} \left[\frac{1}{A_1 A_2} \frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_2} \right] = Z \quad (2.22)$$

If a more accurate equation than Equation (2.22) is required for the plate buckling problem, we can use Equation (2.16) or (2.17) wherein we change from minus to plus the signs in front of the terms involving N_{11} , N_{22} and N_{12} . In these equations, N_{11} , N_{22} and N_{12} represent the stress resultants (acting on a typical plate element) induced by the compressive edge loads ^{and} that give rise to buckling _A are determined from the inplane equilibrium equations, namely, Equations (2.12) and (2.13).

To go back to the large amplitude flexural vibration problem of the plate, besides Equation (2.16) or (2.19), another equation involving W and F has to be obtained. For this, the compatibility equation has to be brought into the picture. The required compatibility equation can be obtained by specialising the compatibility equation of Mushtari-Vlasov approximation of thin shell theory (See the third of Equations (5.1) of Ref(2)). The specialisation required is to let the radii of curvatures r_1 and r_2 of the shell median surface tend to infinity so that the shell median surface flattens out and becomes the median surface of the flat plate. Thus, the compatibility equation can be shown to be:

$$\begin{aligned} & \frac{\partial}{\partial \alpha_1} \left\{ \frac{1}{A_1} \left[\frac{\partial}{\partial \alpha_1} (A_2 \epsilon_2) - \epsilon_1 \frac{\partial A_2}{\partial \alpha_1} - \frac{A_1}{2} \frac{\partial \omega}{\partial \alpha_2} - \omega \frac{\partial A_1}{\partial \alpha_2} \right] \right\} \\ & + \frac{\partial}{\partial \alpha_2} \left\{ \frac{1}{A_2} \left[\frac{\partial}{\partial \alpha_2} (A_1 \epsilon_1) - \epsilon_2 \frac{\partial A_1}{\partial \alpha_2} - \frac{A_2}{2} \frac{\partial \omega}{\partial \alpha_1} - \omega \frac{\partial A_2}{\partial \alpha_1} \right] \right\} = 0 \end{aligned} \quad (2.23)$$

From the stress-strain law for the isotropic plate material and the definition of the stress resultant, it readily follows that

$$\epsilon_1 = \frac{1}{Eh} (N_{11} - \nu N_{22}); \quad \epsilon_2 = \frac{1}{Eh} (N_{22} - \nu N_{11}); \quad \omega = \frac{2(1+\nu)}{Eh} N_{12} \quad (2.24)$$

where the stress resultants N_{11} , N_{22} and N_{12} are defined in

terms of F by Equation (2.20). Upon introducing the above relations into the left hand side of the compatibility equation (2.23), the resulting expression after algebraic simplification and manipulation can be shown to be $\frac{A_1 A_2}{Eh} (\nabla^4 F)$ where,

$\nabla^4(\dots) = \nabla^2 [\nabla^2(\dots)]$ and the Laplacian operator ∇^2 is defined in Equation (2.5) or (2.21). In fact, the expression

$\frac{A_1 A_2}{Eh} (\nabla^4 F)$ can also be obtained by specialising (i.e. let $r_1 \rightarrow \infty$ and $r_2 \rightarrow \infty$) the final equation obtained from the compatibility equation for the case of Mushtari-Vlasov approximation of thin shell theory. (See Section 17, p.84-90 of Ref(2)).

The next step is to introduce ϵ_1 , ϵ_2 and ω as given by Equation (2.3) into the left hand side of the compatibility equation (2.23). In doing this, it helps to remember that ϵ_1 , ϵ_2 and ω as given by Equation (2.3) have a linear part and a nonlinear part. If for the sake of convenience and simplicity, the nonlinear part is neglected for the time being and the resulting linearised relationship for ϵ_1 , ϵ_2 and ω are introduced into the left hand side of Equation (2.23), it will be seen that Equation (2.23) is automatically satisfied. In checking this, use has to be made of the following condition of Gauss specialised for the case of the median surface of the flat plate by letting the radii of curvatures

r_1 and r_2 of the shell median surface tend to infinity:

$$\frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial A_2}{\partial \alpha_1} \right) = - \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial A_1}{\partial \alpha_2} \right) \quad (2.25)$$

If only the nonlinear terms in the strain-displacement relations given by Equation (2.3) are retained and they are introduced into the left hand side of Equation (2.23), the resulting expression will be:

$$\begin{aligned} & \frac{\partial}{\partial \alpha_1} \left\{ \frac{1}{A_1} \left[\frac{\partial}{\partial \alpha_1} \left\{ \frac{A_2}{2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right)^2 \right\} - \frac{1}{2} \left(\frac{\partial A_1}{\partial \alpha_1} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right)^2 \right. \right. \\ & \left. \left. - \frac{1}{A_1 A_2} \left(\frac{\partial w}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \left(\frac{\partial A_1}{\partial \alpha_2} \right) \right] \right\} - \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \left[\frac{1}{A_1 A_2} \left(\frac{\partial w}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \right] \\ & + \frac{\partial}{\partial \alpha_2} \left\{ \frac{1}{A_2} \left[\frac{\partial}{\partial \alpha_2} \left\{ \frac{A_1}{2} \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right)^2 \right\} - \frac{1}{2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right)^2 \right. \right. \\ & \left. \left. - \frac{1}{A_1 A_2} \left(\frac{\partial w}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \left(\frac{\partial A_2}{\partial \alpha_1} \right) \right] \right\} \end{aligned} \quad (2.26)$$

As mentioned earlier, the left hand side of the compatibility equation (2.23) is also equal to $\frac{A_1 A_2}{Eh} (\nabla^4 F)$. Hence,

$$\frac{A_1 A_2}{Eh} (\nabla^4 F) = \text{Expression given by (2.26) above.} \quad (2.27)$$

For working out any practical problem, expression given by (2.26) needs to be simplified. To this end, we employ the same approximation that is involved in making the function L appearing in Equation (2.14) zero, namely, neglect terms that involve products with $\left(\frac{\partial A_1}{\partial \alpha_2}\right)$ or $\left(\frac{\partial A_2}{\partial \alpha_1}\right)$

or $\frac{\partial}{\partial \alpha_1} \left(\frac{A_2}{A_1}\right)$ or $\frac{\partial}{\partial \alpha_2} \left(\frac{A_1}{A_2}\right)$. In this manner, the expression given by (2.26) simplifies to:

$$\frac{1}{A_1 A_2} \left\{ \left(\frac{\partial w}{\partial \alpha_1 \partial \alpha_2} \right)^2 - \left(\frac{\partial^2 w}{\partial \alpha_1^2} \right) \left(\frac{\partial^2 w}{\partial \alpha_2^2} \right) \right\}$$

Thus, Equation (2.27) reduces to:

$$\nabla^4 F = E h \left\{ \left(\frac{1}{A_1 A_2} \frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} \right)^2 - \left(\frac{1}{A_1^2} \frac{\partial^2 w}{\partial \alpha_1^2} \right) \left(\frac{1}{A_2^2} \frac{\partial^2 w}{\partial \alpha_2^2} \right) \right\} \quad (2.28)$$

We can arrive at the right hand side of Equation (2.28) in another way, starting out with a simplified version of the compatibility Equation (2.23) wherein the terms $\left(\epsilon_1 \frac{\partial A_2}{\partial \alpha_1}\right)$

and $\left(\omega \frac{\partial A_1}{\partial \alpha_2}\right)$ are neglected in comparison with the terms

$\left(A_2 \frac{\partial \epsilon_2}{\partial \alpha_1}\right)$ and $\left(\frac{A_1}{2} \frac{\partial \omega}{\partial \alpha_2}\right)$ and similarly, the terms

$\left(\epsilon_2 \frac{\partial A_1}{\partial \alpha_2}\right)$ and $\left(\omega \frac{\partial A_2}{\partial \alpha_1}\right)$ are neglected in comparison

with the terms $\left(A_1 \frac{\partial \epsilon_1}{\partial \alpha_2}\right)$ and $\left(\frac{A_2}{2} \frac{\partial \omega}{\partial \alpha_1}\right)$. The

basis of such a simplification is the same as before, namely, the terms involving the product of the gradients of the Lamé

parameters A_1 and A_2 are neglected in comparison with terms involving the products of the Lamé parameters themselves.

Thus the compatibility Equation (2.23) simplifies to:

$$\frac{A_2}{A_1} \frac{\partial^2 \epsilon_2}{\partial \alpha_1^2} + \frac{A_1}{A_2} \frac{\partial^2 \epsilon_1}{\partial \alpha_2^2} - \frac{\partial^2 \omega}{\partial \alpha_1 \partial \alpha_2} = 0 \quad (2.29)$$

Introducing into the above equation, the nonlinear part of the strain-displacement relations, namely, $\epsilon_1 = \frac{1}{2} \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right)^2$,

$$\epsilon_2 = \frac{1}{2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right)^2 \quad \text{and} \quad \omega = \frac{1}{A_1 A_2} \left(\frac{\partial w}{\partial \alpha_1} \frac{\partial w}{\partial \alpha_2} \right)$$

and neglecting terms involving the products of $\left(\frac{\partial A_1}{\partial \alpha_2} \right), \left(\frac{\partial A_2}{\partial \alpha_1} \right)$ etc., we will get the right hand side of Equation (2.28).

Equation (2.28) represents the most simplified version of the compatibility equation and together with the equilibrium Equation (2.19) constitute a pair of coupled, nonlinear partial differential equations in the two functions W and F . These two equations as they stand are valid for the static case. For the vibration problem, the necessary modification based on the D'Alembert's principle has to be made. Thus, the loading term Z appearing in Equation (2.19) has to be replaced by

$$\left[Z_{\text{applied}}(\alpha_1, \alpha_2, t) - \rho \frac{\partial^2 w}{\partial t^2} \right] \quad \text{where } \rho \text{ is the mass}$$

per unit area of the plate and t is time. For free vibration,

$Z_{\text{applied}}(\alpha_1, \alpha_2, t) = 0.$ **ORIGINAL PAGE IS
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It is worthwhile summarizing the governing equations for free undamped nonlinear flexural vibration of a thin elastic flat plate. These equations in curvilinear orthogonal coordinates α_1 and α_2 are:-

$$\nabla^4 F = Eh \left\{ \left(\frac{1}{A_1 A_2} \frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_2} \right)^2 - \left(\frac{1}{A_1^2} \frac{\partial^2 W}{\partial \alpha_1^2} \right) \left(\frac{1}{A_2^2} \frac{\partial^2 W}{\partial \alpha_2^2} \right) \right\}$$

(compatibility equation) (2.30)

$$\rho \ddot{W} + D \nabla^4 W = \left\{ \frac{N_{11}}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} \right) + \frac{N_{22}}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} \right) + \frac{2}{A_1 A_2} N_{12} \frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_2} \right\}$$

(Equilibrium equation) (2.31)

where, the stress resultants N_{11} , N_{22} and N_{12} are defined in terms of the stress function F by Equation (2.20). However, in view of the approximation involved in Equations (2.30) and (2.31), these stress resultants can be approximately expressed by

$$N_{11} \approx -\frac{1}{A_2^2} \frac{\partial^2 F}{\partial \alpha_2^2} ; \quad N_{22} \approx -\frac{1}{A_1^2} \frac{\partial^2 F}{\partial \alpha_1^2} ; \quad N_{12} \approx \frac{1}{A_1 A_2} \frac{\partial^2 F}{\partial \alpha_1 \partial \alpha_2} \quad (2.32)$$

The above approximate expressions bring to mind the definition of Airy Stress function of two dimensional elasticity. The biharmonic operator ∇^4 is defined by $\nabla^4 = \nabla^2 \cdot \nabla^2$

where, the Laplacian operator ∇^2 is given by

$$\nabla^2(\dots) = \frac{1}{A_1 A_2} \left\{ \frac{\partial}{\partial \alpha_1} \left[\left(\frac{A_2}{A_1} \right) \frac{\partial(\dots)}{\partial \alpha_1} \right] + \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1}{A_2} \right) \frac{\partial(\dots)}{\partial \alpha_2} \right] \right\} \quad (2.33)$$

The above equations can be very readily specialised for the case of any orthogonal system of coordinates α_1 and α_2 ; for example, cartesian, polar, elliptical co-ordinates etc.

$$\text{With } ds_1 = A_1 d\alpha_1 \quad \text{and} \quad ds_2 = A_2 d\alpha_2$$

Equations (2.30) and (2.31) can be written as,

$$\nabla^4 F = E h \left\{ \left(\frac{\partial^2 w}{\partial s_1 \partial s_2} \right)^2 - \left(\frac{\partial^2 w}{\partial s_1^2} \right) \left(\frac{\partial^2 w}{\partial s_2^2} \right) \right\} \quad (2.34)$$

and

$$\rho \ddot{W} + D(\nabla^4 W) = \left\{ N_{11} \frac{\partial^2 w}{\partial s_1^2} + N_{22} \frac{\partial^2 w}{\partial s_2^2} + 2 N_{12} \frac{\partial^2 w}{\partial s_1 \partial s_2} \right\} \quad (2.35)$$

where $\nabla^4 = \nabla^2 \cdot \nabla^2$ and

$$\begin{aligned} \nabla^2(\dots) &= \left\{ \frac{1}{A_2} \frac{\partial}{\partial s_1} \left[A_2 \frac{\partial(\dots)}{\partial s_1} \right] + \frac{1}{A_1} \frac{\partial}{\partial s_2} \left[A_1 \frac{\partial(\dots)}{\partial s_2} \right] \right\} \\ &\approx \left[\frac{\partial^2(\dots)}{\partial s_1^2} + \frac{\partial^2(\dots)}{\partial s_2^2} \right] \end{aligned} \quad (2.36)$$

since the terms involving the products of $\frac{\partial A_2}{\partial \alpha_1}$ and $\frac{\partial A_1}{\partial \alpha_2}$

are neglected in the approximation employed here.

With suitable modifications, the pair of equations (2.30) and (2.31) [or, Equations (2.34) and (2.35)] also represent the governing equations of the problem of post-buckling behavior of thin elastic flat plates. For this, it has to be recalled that both W and F are independent of time t so that the inertia term $\rho \ddot{W} = 0$; also, the signs in front of the terms involving the stress resultants N_{11} , N_{22} and N_{12} defined in terms of the stress function F by Equations (2.20) have to be changed to minus.

Before concluding this section, a brief discussion of the methods of solution involving Equations (2.30) and (2.31) is in order. One method is to assume a deflection function W that satisfies all the boundary conditions of the problem. Based on the assumed W , the compatibility Equation (2.30) is used to determine the corresponding stress function F . In general, the assumed W and the corresponding F will not satisfy exactly the equilibrium Equation (2.31). An approximate solution of this equation involves using the Galerkin's method. The details of this method as applied to nonlinear flexural vibration problems are given in [3] and [4].

An alternate method of solution is based on writing the inplane equilibrium Equations (2.12) and (2.13) in terms of the displacement components u , v and w . For this, it will be recalled that $N_{11} = \frac{Eh}{(1-\nu^2)} (\epsilon_1 + \nu\epsilon_2)$, $N_{22} = \frac{Eh}{(1-\nu^2)} (\epsilon_2 + \nu\epsilon_1)$, and $N_{12} = \frac{Eh}{2(1+\nu)} \omega$, where, ϵ_1 , ϵ_2 and ω are given by Equations (2.3). Thus, Equation (2.12) can be written as:

$$\begin{aligned}
 & \left\{ \frac{\partial}{\partial \alpha_1} \left[\left(\frac{A_2}{A_1} \right) \frac{\partial u}{\partial \alpha_1} \right] - \frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_1} \right)^2 u + \frac{1}{2 A_1} \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1^3}{A_2} \right) \frac{\partial}{\partial \alpha_2} \left(\frac{u}{A_1} \right) \right] \right. \\
 & \quad + \frac{\partial}{\partial \alpha_1} \left[\frac{1}{A_1} \left(\frac{\partial A_1}{\partial \alpha_2} \right) v \right] - \frac{1}{A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \frac{\partial v}{\partial \alpha_2} + \frac{1}{2 A_1} \frac{\partial}{\partial \alpha_2} \left[A_1 A_2 \frac{\partial}{\partial \alpha_1} \left(\frac{v}{A_2} \right) \right] \Big\} \\
 & \quad + \nu \left\{ \frac{\partial}{\partial \alpha_1} \left[\frac{1}{A_1} \left(\frac{\partial A_2}{\partial \alpha_1} \right) u \right] - \frac{1}{A_1} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \frac{\partial u}{\partial \alpha_1} + \frac{\partial^2 v}{\partial \alpha_1 \partial \alpha_2} \right. \\
 & \quad - \frac{1}{2 A_1} \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1^3}{A_2} \right) \frac{\partial}{\partial \alpha_2} \left(\frac{u}{A_1} \right) \right] - \frac{v}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_1} \right) \left(\frac{\partial A_2}{\partial \alpha_1} \right) \\
 & \quad \left. - \frac{1}{2 A_1} \frac{\partial}{\partial \alpha_2} \left[A_1 A_2 \frac{\partial}{\partial \alpha_1} \left(\frac{v}{A_2} \right) \right] \right\} \\
 & = \frac{1}{2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right)^2 - \frac{\partial}{\partial \alpha_1} \left[\left(\frac{A_2}{2} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right)^2 \right] \\
 & \quad - \frac{1}{2 A_1} \frac{\partial}{\partial \alpha_2} \left[\frac{A_1}{A_2} \left(\frac{\partial w}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \right] - \nu \left\{ \frac{A_2}{2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right)^2 \right. \\
 & \quad \left. - \frac{1}{2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right)^2 - \frac{1}{2 A_1} \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1}{A_2} \right) \left(\frac{\partial w}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \right] \right\}
 \end{aligned}$$

In a similar manner, Equation (2.13) can also be written in terms of u , v and w . However, it is easier to obtain that equation from Equation (2.37) by interchanging A_1 and A_2 , α_1 and α_2 and interchanging u and v . Having written the two inplane equilibrium equations in terms of u , v and w , the next step is to determine from these two equations, the u and v functions for an assumed W . The details of this method of approach are given in [5] .

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Section 3.

In this section, the governing equations for the large amplitude flexural vibration of thin elastic shells of revolution are derived. These equations can then be specialised for circular cylindrical, conical, spherical, ogival, torus shells etc., which are all special cases of the shell of revolution.

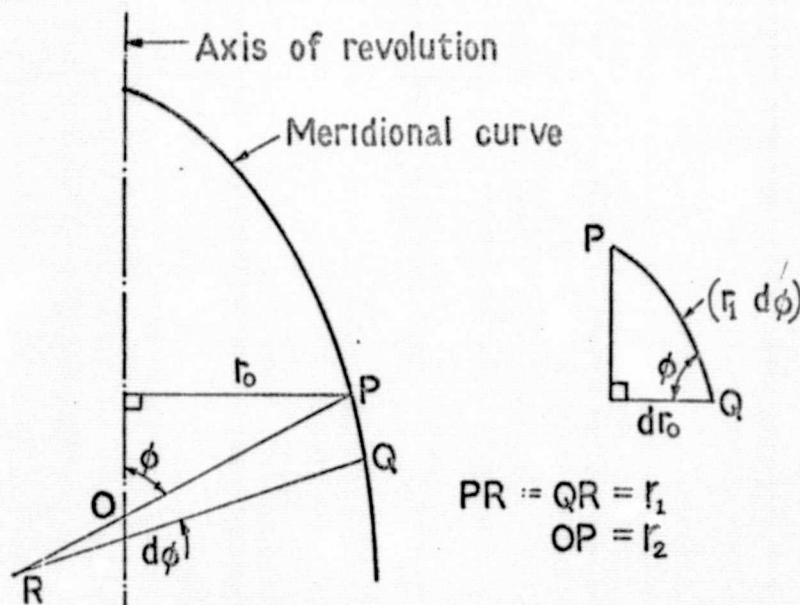


Fig.1

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Figure I shows the meridional curve which when rotated about the axis of revolution generates the shell of revolution. The two principal radii of curvatures of the

shell median surface are r_1 and r_2 and their directions at any point of the shell median surface are assumed to coincide with the directions of the orthogonal co-ordinates ϕ and θ . For a shell of revolution, r_1 and r_2 are functions of ϕ only and not of θ . Also, as can be seen from Figure I,

$$r_0 = r_2 \sin \phi \quad (3.1)$$

and,

$$\frac{dr_0}{d\phi} = r_1 \cos \phi \quad (3.2)$$

The material law for the orthotropic case is given by

$$\sigma_{11} = E_{11} \epsilon_{11} + E_{12} \epsilon_{22}$$

$$\sigma_{22} = E_{22} \epsilon_{22} + E_{12} \epsilon_{11}$$

$$\tau_{12} = G \gamma_{12} \quad (3.3)$$

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where E_{11} , E_{22} , E_{12} and G are four independent material constants of the orthotropic material. For the isotropic case, $E_{11} = E_{22} = \frac{E}{(1-\nu^2)}$, $G = \frac{E}{2(1+\nu)}$ and $E_{12} = \frac{E\nu}{(1-\nu^2)}$

The strain at a general point (ϕ, θ, r) can be written in terms of the six median surface deformational quantities ϵ_1 , ϵ_2 , ω , κ_{11} , κ_{22} and κ_{12} as follows:

$$\epsilon_{11} = \epsilon_1 + r \kappa_{11}$$

$$\epsilon_{22} = \epsilon_2 + r \kappa_{22}$$

$$\gamma_{12} = \omega + 2r \kappa_{12}$$

(3.4)

where, the $\phi - \theta - \gamma$ co-ordinates form as usual a right handed co-ordinate system. The γ co-ordinate is positive in the direction of the unit normal outward as in [2].

The median surface deformation characterised by the extensional strains ϵ_1 and ϵ_2 and the shear strain ω as well as the curvatures κ_{11} , κ_{22} and κ_{12} are defined in terms of the displacement components u , v and w (which are in the ϕ , θ and γ co-ordinate directions respectively) as follows [1], [2] :-

$$\epsilon_1 = \frac{1}{r_1} u_{,\phi} + \frac{w}{r_1} + \frac{1}{2} \left(\frac{1}{r_1} w_{,\phi} \right)^2$$

$$\epsilon_2 = \frac{1}{r_0} v_{,\theta} + \frac{u}{r_2} \cot \phi + \frac{w}{r_2} + \frac{1}{2} \left(\frac{1}{r_0} w_{,\theta} \right)^2$$

$$\omega = \frac{1}{r_1} v_{,\phi} + \frac{1}{r_0} u_{,\theta} - \frac{v}{r_2} \cot \phi + \frac{(w_{,\theta})(w_{,\phi})}{r_1 r_0}$$

$$\kappa_{11} = -\frac{1}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} w_{,\phi} \right)$$

$$\kappa_{22} = -\frac{1}{r_0^2} w_{,\theta\theta} - \frac{1}{r_1 r_2} (w_{,\phi}) \cot \phi$$

$$\kappa_{12} = -\frac{1}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} w_{,\theta} \right) = \frac{1}{r_0} \left[(w_{,\theta}) \frac{\cos \phi}{r_0} - \frac{1}{r_1} w_{,\phi\theta} \right] \quad (3.5)$$

It is worth noting that the expressions for ϵ_1 , ϵ_2 and ω are nonlinear whereas those for κ_{11} , κ_{22} and κ_{12} are linear and dependant only upon w and its derivatives,

but not on u or v . It is easily seen from Equations (3.5) that

$$(\mathcal{K}_{11} + \mathcal{K}_{22}) = -\nabla^2 W = -\frac{1}{r_1 r_0} \left\{ \frac{\partial}{\partial \phi} \left[\left(\frac{r_0}{r_1} \right) v_{,\phi} \right] + \left(\frac{r_1}{r_0} \right) w_{,\theta\theta} \right\} \quad (3.6)$$

The expressions for strain energy U , extensional strain energy U_1 and the bending strain energy U_2 are the same as those given by Equations (2.6), (2.7), (2.8) and (2.9) where, the elemental area dA in the case of the shell of revolution is given by

$$dA = r_0 r_1 d\phi d\theta \quad (3.7)$$

The expression for work done by the applied surface force components Φ , Θ and Z (per unit area) in the co-ordinate directions ϕ , θ and γ respectively is

$$W = \int_{\text{Area}} (\Phi u + \Theta v + Z w) dA$$

Using the principle of minimum total potential, namely,

$\delta(u-w) = \delta U_{\text{tot.}} = 0$, the three equations of equilibrium in the ϕ , θ and γ co-ordinate directions are obtained in terms of the stress resultants N_{ij} and the moment

resultants M_{ij} which are defined as follows:

$$\begin{aligned} N_{ij} &= \int_{-h/2}^{h/2} \sigma_{ij} d\eta \\ M_{ij} &= \int_{-h/2}^{h/2} \eta \sigma_{ij} d\eta \end{aligned} \quad (3.8)$$

Thus,

$$N_{11} = C(\epsilon_1 + \nu \epsilon_2); \quad N_{22} = C(\epsilon_2 + \nu \epsilon_1); \quad N_{12} = \frac{C(1-\nu)}{2} \omega \quad (3.9)$$

and,

$$M_{11} = D(\kappa_{11} + \nu \kappa_{22}); \quad M_{22} = D(\kappa_{22} + \nu \kappa_{11}); \quad M_{12} = D(1-\nu) \kappa_{12} \quad (3.10)$$

where,

$$C = \frac{Eh}{(1-\nu^2)} \quad \text{is the extensional rigidity of the shell}$$

$$\text{and } D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{is the bending rigidity of the shell.}$$

The three equations of equilibrium are:

$$\frac{\partial}{\partial \phi} (r_0 N_{11}) + \frac{\partial}{\partial \theta} (r_1 N_{12}) - (r_1 \cos \phi) N_{22} + r_0 r_1 \Phi = 0 \quad (3.11)$$

$$\frac{\partial}{\partial \theta} (r_1 N_{22}) + \frac{\partial}{\partial \phi} (r_0 N_{12}) + (r_1 \cos \phi) N_{12} + r_0 r_1 \Theta = 0 \quad (3.12)$$

$$\begin{aligned}
 D(\nabla^4 W) - D(1-\nu^2) \left\{ \frac{1}{r_1 r_0^2} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} W_{,\phi\theta\theta} \right) + \frac{1}{r_0 r_1} \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} W_{,\phi\theta} \right) \right] \right. \\
 - \frac{1}{r_0 r_1} \frac{\partial}{\partial \phi} \left[\left(\frac{\sin \phi}{r_1^2} \right) W_{,\phi} \right] + \frac{2}{r_0^2} \left(\frac{\cos^2 \phi}{r_1^2} \right) W_{,\theta\theta} \\
 \left. - 2 \left(\frac{\cos \phi}{r_1 r_0^3} \right) W_{,\phi\theta\theta} + \frac{2}{r_0 r_1} \frac{\partial}{\partial \phi} \left[\left(\frac{\cos \phi}{r_0^2} \right) W_{,\theta\theta} \right] - \frac{2}{r_0 r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0 r_1} W_{,\phi\theta\theta} \right) \right\} \\
 + \left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) - \frac{1}{r_0 r_1} \frac{\partial}{\partial \phi} \left[\frac{r_0}{r_1} N_{11} W_{,\phi} \right] - \frac{1}{r_0 r_1} \frac{\partial}{\partial \theta} \left[\frac{r_1}{r_0} N_{22} W_{,\theta} \right] \\
 - \frac{1}{r_0 r_1} \frac{\partial}{\partial \phi} (N_{12} W_{,\theta}) - \frac{1}{r_0 r_1} \frac{\partial}{\partial \theta} (N_{12} W_{,\phi}) - Z = 0
 \end{aligned}
 \tag{3.13}$$

In terms of the stress and moment resultants, the last equation can also be written as:

$$\begin{aligned}
 \frac{\partial}{\partial \phi} (M_{22} \cos \phi) - \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial}{\partial \phi} (r_0 M_{11}) \right] - \frac{r_1}{r_0} \frac{\partial^2 M_{22}}{\partial \theta^2} - 2 \left(\frac{r_1}{r_0} \right) (\cos \phi) \frac{\partial M_{12}}{\partial \theta} \\
 - 2 \frac{\partial^2 M_{12}}{\partial \phi \partial \theta} + r_0 r_1 \left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) - \frac{\partial}{\partial \phi} \left[\left(\frac{r_0}{r_1} \right) N_{11} W_{,\phi} \right] - \frac{\partial}{\partial \theta} \left[\left(\frac{r_1}{r_0} \right) N_{22} W_{,\theta} \right] \\
 - \frac{\partial}{\partial \phi} (N_{12} W_{,\theta}) - \frac{\partial}{\partial \theta} (N_{12} W_{,\phi}) - r_0 r_1 Z = 0
 \end{aligned}
 \tag{3.14}$$

It is of interest to compare Equations (3.11) and (3.12) which represent the inplane equilibrium equations

with those of the Mushtari-Vlasov approximation of thin shell theory specialised for the case of a shell of revolution so that we let $A_1 = r_1$, $A_2 = r_0$, $\alpha_1 = \phi$ and $\alpha_2 = \theta$ (See Section 17, p.84 of Ref.(2)). This permits the use of the auxiliary function (stress function) $F(\phi, \theta)$ of the Mushtari-Vlasov approximation so that the stress resultants N_{11} , N_{22} and N_{12} are defined in terms of $F(\phi, \theta)$ as follows [2] :

$$\begin{aligned} N_{11} &= -\frac{1}{r_0^2} F_{,\theta\theta} - \frac{\cot\phi}{r_1 r_2} F_{,\phi} \\ N_{22} &= -\frac{1}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} F_{,\phi} \right) \\ N_{12} &= +\frac{1}{r_1} \left\{ \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} F_{,\theta} \right) \right\} \end{aligned} \quad (3.15)$$

It follows from the above that

$$(N_{11} + N_{22}) = -\nabla^2 F \equiv -\frac{1}{r_0 r_1} \left\{ \frac{\partial}{\partial \phi} \left[\left(\frac{r_0}{r_1} \right) F_{,\phi} \right] + \frac{r_1}{r_0} F_{,\theta\theta} \right\} \quad (3.16)$$

Since r_0 , r_1 and r_2 are independent of the θ co-ordinate in the case of shells of revolution, if N_{11} , N_{22} and N_{12} as defined by Equations (3.15) are introduced into the inplane equilibrium equations (3.11) and (3.12), the resulting equations will be:

$$\begin{aligned} \frac{1}{r_1 r_2} \left(\frac{1}{r_1} F_{,\phi} \right) + \Phi &= 0 \\ \frac{1}{r_1 r_2} \left(\frac{1}{r_0} F_{,\theta} \right) + \Theta &= 0 \end{aligned}$$

The above two equations are satisfied exactly for shells of revolution of zero Gaussian curvature (i.e. $\frac{1}{r_1 r_2} = 0$) as in the case of circular cylindrical or conical shells provided the applied load components $\bar{\Phi}$ and $\bar{\Theta}$ are zero. In the case of other shell geometries, the last two equations will not be satisfied exactly and in such cases, the error involved particularly when $\bar{\Phi}$ and $\bar{\Theta}$ are zero, will be assumed to be negligible.

It is worth noting that if in Equations (3.15), the function F is replaced by w , the resulting expressions become that of κ_{22} , κ_{11} and $(-\kappa_{12})$ as given by equations (3.5). From Equations (3.15) it is seen that

$$\left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) = -\bar{D}(F) \quad (3.17)$$

where the operator \bar{D} stands for

$$\bar{D}(\dots) \equiv \frac{1}{r_1 r_0} \left\{ \frac{\partial}{\partial \phi} \left[\frac{r_0}{r_1 r_2} \frac{\partial(\dots)}{\partial \phi} \right] + \frac{1}{r_0} \frac{\partial^2(\dots)}{\partial \theta^2} \right\} \quad (3.18)$$

Similarly, from Equations (3.5), it follows that

$$\left(\frac{\kappa_{11}}{r_2} + \frac{\kappa_{22}}{r_1} \right) = -\bar{D}(w)$$

Because of Equations (3.15) and (3.17), Equation (3.13) which involves the functions $w(\phi, \theta)$ and $F(\phi, \theta)$

can be written as:

$$\begin{aligned}
 D \nabla^4 W - \bar{D}(F) - \frac{1}{r_0 r_1} \left[\frac{\partial}{\partial \phi} \left(\frac{r_0}{r_1} N_{11} W_{,\phi} \right) + \frac{\partial}{\partial \theta} \left(\frac{r_1}{r_0} N_{22} W_{,\theta} \right) \right] \\
 - \frac{1}{r_0 r_1} \frac{\partial}{\partial \phi} \left\{ \frac{W_{,\theta}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} F_{,\theta} \right) \right\} - \frac{1}{r_0 r_1} \frac{\partial}{\partial \theta} \left\{ \frac{W_{,\phi}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} F_{,\theta} \right) \right\} \\
 = Z + D(1-\gamma) \left\{ \frac{1}{r_1 r_0^2} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} W_{,\phi\theta\theta} \right) + \frac{1}{r_1 r_0} \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} W_{,\theta\theta} \right) \right] \right. \\
 - \frac{1}{r_1 r_0} \frac{\partial}{\partial \phi} \left(\frac{\sin \phi}{r_1^2} W_{,\phi} \right) + \frac{2}{r_0^2} \left(\frac{\cos \phi}{r_0} \right)^2 W_{,\theta\theta} - 2 \left(\frac{\cos \phi}{r_1 r_0^3} \right) W_{,\phi\theta\theta} \\
 \left. + \frac{2}{r_0 r_1} \frac{\partial}{\partial \phi} \left[\left(\frac{\cos \phi}{r_0^2} \right) W_{,\theta\theta} \right] - \frac{2}{r_0 r_1} \frac{\partial}{\partial \phi} \left[\frac{1}{r_0 r_1} W_{,\phi\theta\theta} \right] \right\} \quad (3.19)
 \end{aligned}$$

The last equation involving the functions W and F is quite complicated because of the presence of the terms within the brace $\{\dots\}$ on the right hand side of this equation. These terms multiplied by $D(1-\gamma)$ can be shown to stem from the integral $\int_A (\kappa_{11} \cdot \kappa_{22} - \kappa_{12}^2) dA$ appearing in the expression for bending strain energy U_2 as given by Equation (2.8). If the integral mentioned above is neglected so that the bending strain energy is approximated by

$$\left(\frac{D}{2} \right) \int_A (\kappa_{11} + \kappa_{22})^2 dA = \left(\frac{D}{2} \right) \int_A (\nabla^2 W)^2 dA, \quad \text{then,}$$

a considerable simplification of Equation (3.19) results.

Consequently, the following simplified version of the equation of equilibrium in the η coordinate direction is obtained.

$$\begin{aligned} D \nabla^4 W - \bar{D}(F) - \frac{1}{r_0 r_1} \left\{ \frac{\partial}{\partial \phi} \left(\frac{r_0}{r_1} N_{11} W_{,\phi} \right) \right. \\ \left. + \frac{\partial}{\partial \theta} \left(\frac{r_1}{r_0} N_{22} W_{,\theta} \right) - \frac{\partial}{\partial \phi} \left[\frac{W_{,\theta}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} F_{,\theta} \right) \right] \right. \\ \left. - \frac{\partial}{\partial \theta} \left[\frac{W_{,\phi}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} F_{,\theta} \right) \right] \right\} = Z \end{aligned} \quad (3.20)$$

In the last equation, all terms on its left hand side except the first two terms involve coupling between W and F and hence this equation is coupled. If these terms involving coupling are neglected, the resulting linear equation, namely, $D \nabla^4 W - \bar{D}(F) = Z$ represents the equilibrium equation for a shell of revolution according to the Muskhvili-Vlasov approximation of thin shell theory [2] .

Equation (3.20) can still further be simplified by introducing the inplane equilibrium equations (3.11) and (3.12) wherein the loading terms $\bar{\Phi}$ and $\bar{\Theta}$ are neglected. Thus, from Equation (3.20), the following equation is obtained:

$$\begin{aligned} D \nabla^4 W - \bar{D}(F) - \frac{N_{11}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} W_{,\phi} \right) - N_{22} \left[\frac{\cos \phi}{r_0 r_1} W_{,\phi} + \frac{1}{r_0^2} W_{,\theta\theta} \right] \\ - \frac{2N_{12}}{r_0 r_1} \left[W_{,\theta\phi} - \frac{(r_1 \cos \phi)}{r_0} W_{,\theta} \right] = Z \end{aligned} \quad (3.21)$$

Using the expressions for \mathcal{K}_{11} , \mathcal{K}_{22} and \mathcal{K}_{12} as given by Equations (3.5), the last equation can also be written in the form

$$D\nabla^4 W - \bar{D}(F) + N_{11} \mathcal{K}_{11} + N_{22} \mathcal{K}_{22} + 2N_{12} \mathcal{K}_{12} = Z \quad (3.22)$$

It should be noted that Equation (3.22) is also the governing equation for the buckling of a shell of revolution provided it is recalled that in buckling problems, N_{11} and N_{22} are compressive (and similarly for N_{12}) so that for buckling problems Equation (3.22) becomes

$$D(\nabla^4 W) - \left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) - N_{11} \mathcal{K}_{11} - N_{22} \mathcal{K}_{22} - 2N_{12} \mathcal{K}_{12} = Z \quad (3.23)$$

or, by means of Equations (3.5),

$$\begin{aligned} D(\nabla^4 W) - \left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) + \frac{N_{11}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} W_{,\phi} \right) \\ + N_{22} \left[\frac{1}{r_1^2} W_{,\theta\theta} + \frac{\cot \phi}{r_1 r_2} W_{,\phi} \right] + \frac{2N_{12}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} W_{,\theta} \right) = Z \end{aligned} \quad (3.24)$$

In the above equation, N_{11} , N_{22} and N_{12} represent the stress resultants (acting on a typical shell element) induced by the applied load(s) that give rise to buckling.

Their values just prior to buckling are determined from the inplane equilibrium equations, namely, Equations (3.11) and (3.12).

Because of Equation (3.15), Equation (3.21) represents one equation involving the two functions W and F . A second equation involving W and F is obtained from the compatibility equation. If in Equations (3.5), the nonlinear terms are dropped, the resulting expressions are exactly those of the Mushtari-Vlasov approximation specialised for shells of revolution. The compatibility equation that is made use of in the Mushtari-Vlasov approximation specialised for shells of revolution can be shown to be [2] :

$$\begin{aligned} r_0 r_1 \left(\frac{\mathcal{U}_{11}}{r_2} + \frac{\mathcal{U}_{22}}{r_1} \right) + \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial}{\partial \phi} (r_0 \epsilon_2) \right] - \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} \frac{dr_0}{d\phi} \epsilon_1 \right) \\ + \frac{r_1}{r_0} \frac{\partial^2 \epsilon_1}{\partial \theta^2} - \left(\frac{r_1}{r_0} \right) (\cos \phi) \frac{\partial \omega}{\partial \theta} - \frac{\partial^2 \omega}{\partial \phi \partial \theta} = 0 \end{aligned} \quad (3.25)$$

If we linearise the expressions for ϵ_1 , ϵ_2 and ω as given by Equation (3.5) and introduce the resulting expressions into the left hand side of Equation (3.25), the result is

$$\left\{ - \left[\frac{\partial}{\partial \phi} \frac{1}{r_1} \sin \phi \right] + \frac{\mathcal{U}_{2\theta}}{r_2} + r_0 r_1 \bar{D}(w) \right\} + r_0 r_1 \left(\frac{\mathcal{U}_{11}}{r_2} + \frac{\mathcal{U}_{22}}{r_1} \right) \}$$

The u and v dependant terms arise because in writing the

expressions for curvatures κ_{11} , κ_{22} and κ_{12} as dependant solely on the displacement component $w(\phi, \theta)$, [see Equation (3.5)], an approximation is involved and this approximation is also made use of in the Muskhvili-Vlasov approximation. If the u and v dependant terms in the curvature expressions had been kept, the terms $-\left[\frac{\partial}{\partial \phi} - \left(\frac{u}{r_1} \sin \phi\right) + \frac{v}{r_2}\right]$ would not arise. In what follows, these terms depending upon u and v will be neglected. In this approximate sense, the right hand side of Equation (3.25), upon introducing the relations for ϵ_1 , ϵ_2 and ω as given by Equation (3.5), will become

$$\left[-r_0 r_1 \bar{D}(W) + r_0 r_1 \left(\frac{\kappa_{11}}{r_2} + \frac{\kappa_{22}}{r_1}\right)\right] \quad \text{plus nonlinear terms depending upon } W \text{ and its derivatives .}$$

From Equations (3.9) it follows that

$$\epsilon_1 = \frac{1}{Eh} (N_{11} - \gamma N_{22}) ; \quad \epsilon_2 = \frac{1}{Eh} (N_{22} - \gamma N_{11}) ; \quad \omega = \frac{N_{12}}{Gh} \quad (3.26)$$

Equations (3.15) and (3.26) yield

$$\begin{aligned} Eh \epsilon_1 &= -\frac{1}{r_0^2} F_{,\theta\theta} - \frac{\cot \phi}{r_1 r_2} F_{,\phi} + \frac{\gamma}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} F_{,\phi} \right) \\ Eh \epsilon_2 &= -\frac{1}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} F_{,\phi} \right) + \frac{\gamma}{r_0^2} F_{,\theta\theta} + \frac{\gamma \cot \phi}{r_1 r_2} F_{,\phi} \\ Eh \omega &= 2(1+\gamma) \left[\frac{1}{r_0 r_1} F_{,\phi\theta} - \frac{\cot \phi}{r_0^2} F_{,\theta} \right] \end{aligned}$$

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Introducing Equation (3.27) into the left hand side of Equation (3.25) the following expression is obtained:-

$$\begin{aligned}
 & \frac{1}{Eh} \left\{ \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial}{\partial \phi} \left[-\frac{r_0}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} F_{,\phi} \right) + \frac{\gamma}{r_0} F_{,\theta\theta} + \frac{\gamma \cos \phi}{r_1} F_{,\phi} \right] \right\} \right. \\
 & + \frac{\partial}{\partial \phi} \left[\frac{\cos \phi}{r_0^2} F_{,\theta\theta} + \frac{\cos^2 \phi}{r_1 r_0} F_{,\phi} - \frac{\gamma \cos \phi}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} F_{,\phi} \right) \right] \\
 & - \frac{r_1}{r_0} \left[\frac{1}{r_0^2} F_{,\theta\theta\theta\theta} + \frac{\cot \phi}{r_1 r_2} (F_{,\phi\theta\theta}) - \frac{\gamma}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} F_{,\phi\theta\theta} \right) \right] \\
 & - 2 \left(\frac{r_1 \cos \phi}{r_0} \right) (1 + \gamma) \left[\frac{1}{r_0 r_1} (F_{,\phi\theta\theta}) - \left(\frac{\cos \phi}{r_0^2} \right) F_{,\theta\theta} \right] \\
 & \left. - 2(1 + \gamma) \frac{\partial}{\partial \phi} \left[\frac{1}{r_0 r_1} (F_{,\phi\theta\theta}) - \left(\frac{\cos \phi}{r_0^2} \right) F_{,\theta\theta} \right] \right\} + r_0 r_1 \left(\frac{\mathcal{H}_{11}}{r_2} + \frac{\mathcal{H}_{22}}{r_1} \right) \quad (3.28)
 \end{aligned}$$

The above expression can be shown to be equal to

$$\left[\frac{r_0 r_1}{Eh} (\nabla^4 F) + r_0 r_1 \left(\frac{\mathcal{H}_{11}}{r_2} + \frac{\mathcal{H}_{22}}{r_1} \right) \right] \quad \text{where,}$$

$\nabla^4 F \equiv \nabla^2 (\nabla^2 F)$ and the laplacian operator ∇^2 is defined as

$$\nabla^2(\dots) \equiv \frac{1}{r_0 r_1} \left\{ \frac{\partial}{\partial \phi} \left[\frac{r_0}{r_1} \frac{\partial(\dots)}{\partial \phi} \right] + \left(\frac{r_1}{r_0} \right) \frac{\partial^2(\dots)}{\partial \theta^2} \right\}. \quad (3.28a)$$

Introducing ϵ_1 , ϵ_2 and ω as given by Equation (3.5) into the left hand side of Equation (3.25) and

neglecting, as mentioned earlier, the terms

$$\left[-\frac{\partial}{\partial \phi} \left(\frac{u}{r_1} \sin \phi \right) - \frac{v_{, \theta}}{r_2} \right]$$

, the following

expression is obtained:

$$\begin{aligned} & -r_0 r_1 \bar{D}(w) + \left\{ \left(\frac{r_0}{r_1} \right) \left[\frac{\partial}{\partial \phi} \left(\frac{1}{r_0} w_{, \theta} \right) \right]^2 - \frac{(w_{, \theta \theta})(w_{, \phi \phi})}{r_0 r_1} \right\} \\ & + \left\{ \left(\frac{r_0}{r_1} \right) \left(\frac{1}{r_0} w_{, \theta} \right) \frac{\partial^2}{\partial \phi^2} \left(\frac{1}{r_0} w_{, \theta} \right) + \left(\frac{1}{r_0} w_{, \theta} \right) \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} w_{, \theta} \right) \frac{\partial}{\partial \phi} \left(\frac{r_0}{r_1} \right) \right. \\ & + \left(\frac{\sin \phi}{2} \right) \left(\frac{1}{r_1} w_{, \phi} \right)^2 - (w_{, \theta \theta})(w_{, \phi}) \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} \right) \\ & - (\cos \phi) \left[\left(\frac{1}{r_1} w_{, \phi} \right) \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} w_{, \phi} \right) + \frac{1}{r_0^2} (w_{, \phi})(w_{, \theta \theta}) \right. \\ & \left. \left. + \frac{1}{r_0^2} (w_{, \theta})(w_{, \theta \phi}) \right] \right\} + r_0 r_1 \left(\frac{x_{11}}{r_2} + \frac{x_{22}}{r_1} \right) \end{aligned}$$

(3.29)

On equating expressions (3.28) and (3.29) the following compatibility equation is obtained:

$$\begin{aligned} \frac{1}{Eh} (\nabla^4 F) + \bar{D}(w) &= \frac{1}{r_0 r_1} \left\{ \left(\frac{r_0}{r_1} \right) \left[\frac{\partial}{\partial \phi} \left(\frac{1}{r_0} w_{, \theta} \right) \right]^2 \right. \\ &- \frac{(w_{, \theta \theta})(w_{, \phi \phi})}{r_0 r_1} + \left\{ \left(\frac{r_0}{r_1} \right) \left(\frac{1}{r_0} w_{, \theta} \right) \frac{\partial^2}{\partial \phi^2} \left(\frac{1}{r_0} w_{, \theta} \right) \right. \\ &+ \left(\frac{1}{r_0} w_{, \theta} \right) \frac{\partial}{\partial \phi} \left(\frac{1}{r_0} w_{, \theta} \right) \frac{\partial}{\partial \phi} \left(\frac{r_0}{r_1} \right) + \left(\frac{\sin \phi}{2} \right) \left(\frac{1}{r_1} w_{, \phi} \right)^2 \\ &- (w_{, \theta \theta})(w_{, \phi}) \frac{\partial}{\partial \phi} \left(\frac{1}{r_0 r_1} \right) - \cos \phi \left[\left(\frac{1}{r_1} w_{, \phi} \right) \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} w_{, \phi} \right) \right. \\ &\left. \left. + \frac{1}{r_0^2} (w_{, \phi})(w_{, \theta \theta}) + \frac{1}{r_0^2} (w_{, \theta})(w_{, \theta \phi}) \right] \right\} \end{aligned}$$

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(3.30)

In the above equation, the right hand side consists of nonlinear terms and if they are neglected, the resulting equation is that corresponding to the Mushtari-Vlasov approximation of thin shell theory [2] .

Equation (3.30) is too complicated and a simplification of it is essential if any solution is to be found for an actual problem. In such a simplification - to be justified later - all the nonlinear terms contained within the brace {...} on the right hand side of Equation (3.30) are neglected. If in the remaining two nonlinear terms, we further approximate and write $\frac{\partial}{\partial \phi} \left(\frac{1}{r_0} W, \theta \right) \approx \frac{1}{r_0} W, \theta \phi$ which means neglecting the term $\left[- \frac{W, \theta}{r_0^2} \left(\frac{dr_0}{d\phi} \right) \right]$ in comparison to $\left(\frac{1}{r_0} W, \theta \phi \right)$, Equation (3.30) will finally simplify to:

$$\nabla^4 F + Eh \left[\bar{D}(W) \right] = Eh \left\{ \frac{1}{r_0^2 r_1^2} (W, \theta, \theta)^2 - \left(\frac{1}{r_0^2} W, \theta \theta \right) \left(\frac{1}{r_1^2} W, \phi \phi \right) \right\} \quad (3.31)$$

The basis of the simplification mentioned above requires justification. To this end, the compatibility equation (3.25) is re-written as:

$$\begin{aligned} \frac{\partial}{\partial \phi} \left[\frac{1}{r_1} \frac{\partial}{\partial \phi} (r_0 \epsilon_2) \right] - \frac{\partial}{\partial \phi} \left(\frac{\epsilon_1}{r_1} \frac{dr_0}{d\phi} \right) + \frac{r_1}{r_0} \frac{\partial^2 \epsilon_1}{\partial \theta^2} \\ - \frac{1}{r_0} \left(\frac{dr_0}{d\phi} \right) \frac{\partial \omega}{\partial \theta} - \frac{\partial^2 \omega}{\partial \phi \partial \theta} = -r_0 r_1 \left(\frac{\kappa_{11}}{r_2} + \frac{\kappa_{22}}{r_1} \right) \end{aligned}$$

If in the above equation, terms involving products with $\left(\frac{dr_0}{d\phi} \right)$ are neglected, the resulting simplified compatibility

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equation is:

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$$\frac{r_0}{r_1} \frac{\partial^2 \epsilon_2}{\partial \phi^2} + \frac{r_1}{r_0} \frac{\partial^2 \epsilon_1}{\partial \theta^2} - \frac{\partial^2 \omega}{\partial \phi \partial \theta} = -r_0 r_1 \left(\frac{\mathcal{K}_{11}}{r_2} + \frac{\mathcal{K}_{22}}{r_1} \right)$$

If in the above equation, the nonlinear part of the expressions for ϵ_1 , ϵ_2 and ω as given by Equation (3.5) are introduced, the left hand side of the last equation yields:

$$\left\{ \frac{r_0}{2r_1} \frac{\partial^2}{\partial \phi^2} \left[\left(\frac{1}{r_0} W_{,\theta} \right)^2 \right] + \frac{r_1}{2r_0} \frac{\partial^2}{\partial \theta^2} \left[\left(\frac{1}{r_1} W_{,\phi} \right)^2 \right] - \frac{\partial^2}{\partial \phi \partial \theta} \left[\left(\frac{1}{r_0} W_{,\theta} \right) \left(\frac{1}{r_1} W_{,\phi} \right) \right] \right\}$$

On expanding the above expression and neglecting terms involving products with $\left(\frac{dr_0}{d\phi} \right)$ or $\left(\frac{dr_1}{d\phi} \right)$, we will finally obtain the expression,

$$\frac{1}{r_0 r_1} \left[\left(W_{,\theta\phi} \right)^2 - \left(W_{,\theta\theta} \right) \left(W_{,\phi\phi} \right) \right]$$

The above expression multiplied by $\frac{Eh}{r_0 r_1}$ is the right hand side of Equation (3.31).

Finally then, we have Equations (3.21) and (3.31) as the governing equations (for the static case) for the nonlinear analysis of thin elastic shells of revolution.

They are a pair of coupled, nonlinear partial differential equations. If the nonlinear terms are neglected, Equations (3.21) and (3.31) reduce to the governing equations of Mushtari-Vlasov approximation of thin shell theory specialised for the case of shells of revolution.

It is worth mentioning that Equations (3.21) and (3.31) also represent the governing equations of the problem of postbuckling behavior of thin elastic shells of revolution provided the signs in front of the terms involving the stress resultants N_{11} , N_{22} and N_{12} defined in terms of the stress function $F(\phi, \theta)$ by Equation (3.15) are changed from minus to plus. Thus Equation (3.24) holds good; only, the stress resultants are to be determined from Equations (3.31) and (3.15).

For the analyses of nonlinear flexural vibration problems of thin elastic shells of revolution, Equation (3.31) remains unchanged as it represents the compatibility equation. However, in Equation (3.21) Z has to be replaced by

$\left[Z_{\text{applied}}(\phi, \theta, t) - \rho \frac{\partial^2 w}{\partial t^2} \right]$ where, ρ is the mass per unit area of the shell and t is time. For free vibration problems, $Z(\phi, \theta, t) = 0$.

It is worthwhile summarising the governing equations for free undamped nonlinear flexural vibration of thin elastic

shells of revolution. These equations (in orthogonal co-ordinates ϕ and θ) involving the normal displacement function $W(\phi, \theta, t)$ and the auxiliary function (stress function) $F(\phi, \theta, t)$ are:-

$$\nabla^4 F = Eh \left\{ -\bar{D}(W) + \left(\frac{1}{r_1 r_0} W_{,\theta\phi} \right)^2 - \left(\frac{1}{r_2^2} W_{,\theta\theta} \right) \left(\frac{1}{r_1^2} W_{,\phi\phi} \right) \right\} \quad (3.32)$$

(compatibility equation)

$$\rho \ddot{W} + D(\nabla^4 W) = \left\{ \bar{D}(F) + \frac{N_{11}}{r_1} \frac{\partial}{\partial \phi} \left(\frac{1}{r_1} W_{,\phi} \right) + N_{22} \left[\frac{\cos \phi}{r_0 r_1} W_{,\phi} + \frac{1}{r_0^2} W_{,\theta\theta} \right] + 2N_{12} \left[\frac{1}{r_0 r_1} W_{,\theta\phi} - \frac{\cos \phi}{r_0} \left(\frac{1}{r_0} W_{,\theta} \right) \right] \right\} \quad (3.33)$$

(Equilibrium equation)

The stress resultants N_{11} , N_{22} and N_{12} are defined in terms of the stress function F by equation (3.15). However, in view of the approximations involved in Equations (3.32) and (3.33), these stress resultants can be approximately expressed by

$$N_{11} \approx -\frac{1}{r_0^2} F_{,\theta\theta} ; \quad N_{22} \approx -\frac{1}{r_1^2} F_{,\phi\phi} ; \quad N_{12} \approx \frac{1}{r_1 r_0} F_{,\phi\theta}$$

Also, the two terms in Equation (3.33) involving

$$\frac{\cos \phi}{r_0} = \frac{1}{r_0 r_1} \left(\frac{dr_0}{d\phi} \right) \quad \text{can be dropped.}$$

The biharmonic operator ∇^4 is defined by $\nabla^4 = \nabla^2 \cdot \nabla^2$ where, the Laplacian operator ∇^2 is defined by Equation (3.28a). But in view of the approximation employed in obtaining Equations (3.32), (3.33), N_{11} and so on, ∇^2 can be approximated as

$$\nabla^2(\dots) = \left[\frac{1}{r_1^2} \frac{\partial^2(\dots)}{\partial \phi^2} + \frac{1}{r_0^2} \frac{\partial^2(\dots)}{\partial \theta^2} \right]$$

The above equations can be readily specialised for the case of any shell of revolution; for example, circular cylindrical, conical, ogival, torus, spherical shell and so on.

With $ds_1 = r_1 d\phi$ and $ds_2 = r_0 d\theta$, Equations (3.32) and (3.33) can also be written as:

$$\nabla^4 F = Eh \left\{ -\bar{D}(w) + \left(\frac{\partial^2 w}{\partial s_1 \partial s_2} \right)^2 - \left(\frac{\partial^2 w}{\partial s_1^2} \right) \left(\frac{\partial^2 w}{\partial s_2^2} \right) \right\} \quad (3.34)$$

and,

$$\begin{aligned} \rho \ddot{W} + D(\nabla^4 W) = & \left\{ \bar{D}(F) + N_{11} \frac{\partial^2 W}{\partial s_1^2} + N_{22} \left[\frac{1}{r_0} \left(\frac{dr_0}{ds_1} \right) \frac{\partial W}{\partial s_1} + \frac{\partial^2 W}{\partial s_2^2} \right] \right. \\ & \left. + 2N_{12} \left[\frac{\partial^2 W}{\partial s_1 \partial s_2} - \frac{1}{r_0} \left(\frac{dr_0}{ds_1} \right) \frac{\partial W}{\partial s_2} \right] \right\} \end{aligned} \quad (3.35)$$

where, $\nabla^4 = \nabla^2 \cdot \nabla^2$ and from Equation (3.28a)

$$\begin{aligned} \nabla^2(\dots) = & \left\{ \frac{1}{r_0} \frac{\partial}{\partial s_1} \left[r_0 \frac{\partial(\dots)}{\partial s_1} \right] + \frac{\partial^2(\dots)}{\partial s_2^2} \right\} \\ \approx & \left[\frac{\partial^2(\dots)}{\partial s_1^2} + \frac{\partial^2(\dots)}{\partial s_2^2} \right] \end{aligned} \quad (3.36)$$

since terms involving products of $\frac{\partial r_0}{\partial s_1}$ can be neglected in which case the two terms proportional to $(\frac{1}{r_0} \frac{dr_0}{ds_1})$ in Equation (3.35) can also be neglected.

It is of interest to compare Equations (3.34) and (3.35) with Equations (2.34) and (2.35) applicable for the case of thin flat plates. If in Equation (3.34), the term $\bar{D}(W)$ is dropped, the resulting equation, formally at any rate, coincides with Equation (2.24). Similarly, in the absence of the term $\bar{D}(F)$ and the two terms involving the products of $\frac{1}{r_0}(\frac{dr_0}{ds_1})$, Equation (3.35) reduces to Equation (2.35). Thus the influence of the shell geometry is brought in essentially by the terms $\bar{D}(W)$ and $\bar{D}(F)$, where the operator \bar{D} is defined by Equation (3.18).

As outlined at the end of the last section dealing with thin flat plates, here also two methods of solution of Equations (3.32) and (3.33) are possible. One is to assume a deflection function W that satisfies all the boundary conditions of the problem and based on the assumed W , find the stress function F that satisfies the compatibility equation (3.32). In general, the assumed W and the corresponding F will not satisfy exactly the equilibrium equation (3.33). The Galerkin's method is employed to obtain an approximate solution [3] , [4] .

The alternate method is based on writing the inplane equilibrium Equations (3.11) and (3.12) in terms of the displacement components u , v and w so that from the resulting two equations, we can try to find u and v for an assumed W . For example, if in the inplane equilibrium equation (3.11), Equations (3.9) are introduced and the median surface strains ϵ_1 , ϵ_2 and ω defined in terms of u, v and w by Equations (3.5) are made use of, the resulting equation can be shown to be:

$$\begin{aligned}
 & \left[\frac{\partial}{\partial \phi} \left(\frac{r_0}{r_1} u_{,\phi} \right) - \left(\frac{r_1 \cos^2 \phi}{r_0} \right) u + \frac{r_1}{2r_0} u_{,\theta\theta} \right] \\
 & - \gamma \left[u \sin \phi + \frac{r_1}{2r_0} u_{,\theta\theta} \right] + \left[\frac{1}{2} v_{,\phi\theta} - \left(\frac{r_1}{r_0} \right) (\cos \phi) v_{,\theta} \right. \\
 & \left. - \left(\frac{r_1 \cos \phi}{2r_0} \right) v_{,\theta} + \gamma \left[\frac{1}{2} v_{,\theta\phi} - \left(\frac{r_1 \cos \phi}{2r_0} \right) v_{,\theta} \right] \right] \\
 & = \left\{ \left(\frac{r_1 \cos \phi}{r_2} \right) W - \frac{\partial}{\partial \phi} \left(\frac{r_0}{r_1} W \right) - \gamma \frac{\partial}{\partial \phi} \left(\frac{r_0}{r_2} W \right) \right. \\
 & \left. + \gamma (\cos \phi) W + \left(\frac{r_1 \cos \phi}{2} \right) \left(\frac{1}{r_0} W_{,\theta} \right)^2 - \frac{\partial}{\partial \phi} \left[\frac{r_0}{2} \left(\frac{1}{r_1} W_{,\phi} \right)^2 \right] \right. \\
 & \left. - \frac{1}{2r_0} \left[(W_{,\theta\theta}) W_{,\phi} + (W_{,\theta}) (W_{,\phi\theta}) \right] - \frac{\gamma}{2} \frac{\partial}{\partial \phi} \left[r_0 \left(\frac{1}{r_0} W_{,\theta} \right)^2 \right] \right. \\
 & \left. + \frac{\gamma}{2} (r_1 \cos \phi) \left(\frac{1}{r_1} W_{,\phi} \right)^2 + \frac{\gamma}{2r_0} \left[W_{,\phi} W_{,\theta\theta} + W_{,\theta} W_{,\phi\theta} \right] \right\}
 \end{aligned}$$

Similarly, starting out with Equation (3.12), another equation in terms of u , v and w can be written. The right hand sides of these two equations involve W and its derivatives so that from these two equations, we can try to find u and v for an assumed W . This method has been used in the solution of the problem of nonlinear flexural vibration of oval cylindrical shells [5] .

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Section 4.

Having derived the governing equations for the large amplitude (nonlinear) flexural vibration of thin elastic flat plates using curvilinear orthogonal co-ordinates α_1 and α_2 as well as of thin elastic shells of revolution, it is possible to generalise and write down the governing equations for the large amplitude flexural vibration of any thin elastic shell. From these very general equations, the governing equations applicable for shells of revolution or thin flat plates can be obtained as special cases.

For any thin elastic shell, let a typical point on the median surface be P. At P, the directions of the principal radii of curvatures which are orthogonal and the orthogonal co-ordinate directions α_1 and α_2 on the shell median surface are assumed to coincide. The elemental length ds on the shell median surface is given by

$$\begin{aligned}(ds)^2 &= (A_1 d\alpha_1)^2 + (A_2 d\alpha_2)^2 \\ &= (ds_1)^2 + (ds_2)^2\end{aligned}\tag{4.1}$$

where, A_1 and A_2 are the Lamé parameters and are in general functions of α_1 and α_2 . The principal radii of curvatures at a typical point such as P are denoted by r_1 and r_2 and they too are in general functions of α_1 and α_2 . The two Lamé parameters A_1 and A_2 and the two principal radii of

curvatures at a typical point such as P satisfy the three conditions known as the Gauss-Codazzi conditions, namely,

$$\frac{\partial}{\partial \alpha_1} \left(\frac{A_2}{r_2} \right) = \frac{1}{r_1} \left(\frac{\partial A_2}{\partial \alpha_1} \right); \quad \frac{\partial}{\partial \alpha_2} \left(\frac{A_1}{r_1} \right) = \frac{1}{r_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) \quad (4.2)$$

(Conditions of Codazzi)

and,

$$\left[\frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial A_2}{\partial \alpha_1} \right) + \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial A_1}{\partial \alpha_2} \right) \right] = - \frac{A_1 A_2}{r_1 r_2} \quad (4.3)$$

(Condition of Gauss)

The strain at a general point $(\alpha_1, \alpha_2, \gamma)$ can be written in terms of the six median surface deformational quantities $\epsilon_1, \epsilon_2, \omega, \kappa_{11}, \kappa_{22}$ and κ_{12} as in Equation (3.4), where [1] ,

$$\begin{aligned} \epsilon_1 &= \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{v}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) + \frac{w}{r_1} + \frac{1}{2} \left[\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right]^2 \\ \epsilon_2 &= \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{u}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) + \frac{w}{r_2} + \frac{1}{2} \left[\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right]^2 \\ \omega &= \left\{ \frac{1}{A_1} \frac{\partial v}{\partial \alpha_1} - \frac{u}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) + \frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} - \frac{v}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) + \left[\frac{1}{A_1 A_2} \left(\frac{\partial w}{\partial \alpha_1} \right) \left(\frac{\partial w}{\partial \alpha_2} \right) \right] \right\} \\ \kappa_{11} &= - \left[\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) + \frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) \right] \\ \kappa_{22} &= - \left[\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) + \frac{1}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) \right] \\ \kappa_{12} &= \frac{1}{A_1 A_2} \left[- \frac{\partial^2 w}{\partial \alpha_1 \partial \alpha_2} + \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} \right) + \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} \right) \right] \end{aligned} \quad (4.4)$$

The last set of equations specialise to those applicable for the case of thin flat plates, namely, Equations (2.3) and (2.4), provided it is recalled that the radii of curvatures (r_1 and r_2) of the flat plate median surface are infinity. With $r_1 \rightarrow \infty$ and $r_2 \rightarrow \infty$, Equations (4.4) reduce to Equations (2.3) and (2.4). Furthermore, with $A_1 = r_1$ and $A_2 = r_2$ as well as $\alpha_1 = \phi$ and $\alpha_2 = \theta$, Equations (4.4) specialise to those applicable for the case of thin shells of revolution, namely, Equations (3.5). It should also be remembered that for shells of revolution, r_1 and r_2 (and hence $r_2 = \frac{r_0}{\sin \phi}$) are functions of ϕ only and not of θ coordinate.

From Equations (4.4), it follows that

$$(\mathcal{H}_{11} + \mathcal{H}_{22}) = -\nabla^2 W \equiv -\frac{1}{A_1 A_2} \left\{ \frac{\partial}{\partial \alpha_1} \left[\left(\frac{A_2}{A_1} \right) \frac{\partial W}{\partial \alpha_1} \right] + \frac{\partial}{\partial \alpha_2} \left[\left(\frac{A_1}{A_2} \right) \frac{\partial W}{\partial \alpha_2} \right] \right\} \quad (4.5)$$

Also,

$$\left[\left(\frac{\mathcal{H}_{11}}{r_2} \right) + \left(\frac{\mathcal{H}_{22}}{r_1} \right) \right] = -\bar{D}(W) \equiv -\frac{1}{A_1 A_2} \left\{ \frac{\partial}{\partial \alpha_1} \left[\frac{1}{r_2} \left(\frac{A_2}{A_1} \right) \frac{\partial W}{\partial \alpha_1} \right] + \frac{\partial}{\partial \alpha_2} \left[\frac{1}{r_1} \left(\frac{A_1}{A_2} \right) \frac{\partial W}{\partial \alpha_2} \right] \right\} \quad (4.6)$$

It is to be noted that in the case of a spherical shell, $r_1 = r_2 = R$ where R is the radius of the spherical shell so that the operator $\bar{D}(\dots) = \frac{1}{R} \nabla^2(\dots)$

Using Equations (4.4), the energy method and steps analagous to those outlined in the last section leading to

Equations (3.32) and (3.33), the governing equations for the free, undamped, large amplitude (nonlinear) flexural vibration of thin elastic shells can be readily obtained in terms of the normal displacement function $W(\alpha_1, \alpha_2, t)$ and the auxiliary function (stress function) $F(\alpha_1, \alpha_2, t)$. These equations can be shown to be:-

$$\nabla^4 F = Eh \left\{ -\bar{D}(W) + \left(\frac{1}{A_1 A_2} \frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_2} \right)^2 - \left(\frac{1}{A_1^2} \frac{\partial^2 W}{\partial \alpha_1^2} \right) \left(\frac{1}{A_2^2} \frac{\partial^2 W}{\partial \alpha_2^2} \right) \right\} \quad (4.7)$$

(Compatibility equation)

$$\rho \ddot{W} + D(\nabla^4 W) = \left\{ \bar{D}(F) + \frac{N_{11}}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial W}{\partial \alpha_1} \right) + \frac{N_{22}}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial W}{\partial \alpha_2} \right) + \frac{2}{A_1 A_2} (N_{12}) \frac{\partial^2 W}{\partial \alpha_1 \partial \alpha_2} \right\} \quad (4.8)$$

(Equilibrium equation)

where,

$$N_{11} = - \left[\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial F}{\partial \alpha_2} \right) + \frac{1}{A_1 A_2} \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_1} \frac{\partial F}{\partial \alpha_1} \right) \right]$$

$$N_{22} = - \left[\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial F}{\partial \alpha_1} \right) + \frac{1}{A_1 A_2} \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_2} \frac{\partial F}{\partial \alpha_2} \right) \right]$$

and,

$$N_{12} = + \frac{1}{A_1 A_2} \left[\frac{\partial^2 F}{\partial \alpha_1 \partial \alpha_2} - \left(\frac{\partial A_1}{\partial \alpha_2} \right) \left(\frac{1}{A_1} \frac{\partial F}{\partial \alpha_1} \right) - \left(\frac{\partial A_2}{\partial \alpha_1} \right) \left(\frac{1}{A_2} \frac{\partial F}{\partial \alpha_2} \right) \right]$$

However, in view of the approximations involved in Equations (4.7) and (4.8) which are the same as those involved in obtaining Equations (3.32) and (3.33), the stress resultants can be approximated by:

$$\begin{aligned} N_{11} &\approx -\frac{1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{1}{A_2} \frac{\partial F}{\partial \alpha_2} \right) ; \\ N_{22} &\approx -\frac{1}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{1}{A_1} \frac{\partial F}{\partial \alpha_1} \right) ; \\ \text{and,} \\ N_{12} &\approx \frac{1}{A_1 A_2} \frac{\partial^2 F}{\partial \alpha_1 \partial \alpha_2} \end{aligned} \quad \begin{array}{l} \text{ORIGINAL PAGE IS} \\ \text{OF POOR QUALITY} \end{array} \quad (4.9a)$$

The approximation involves neglect of those terms involving products of the gradients of the Lamé parameters, namely $\left(\frac{\partial A_1}{\partial \alpha_2} \right)$, $\left(\frac{\partial A_2}{\partial \alpha_1} \right)$ etc.

It follows from Equations (4.9) that

$$(N_{11} + N_{22}) = -\nabla^2 F \quad \text{and} \quad \left[(N_{11}/r_1) + (N_{22}/r_2) \right] = -\bar{D}(F).$$

The biharmonic operator ∇^4 is defined by $\nabla^4 = \nabla^2 \cdot \nabla^2$ where, the Laplacian operator ∇^2 is given by Equation (4.5).

Equations (4.7) and (4.8) can be readily specialised for the case of thin flat plates by letting $r_1 \rightarrow \infty$ and $r_2 \rightarrow \infty$ so that $\bar{D}(W) = 0$ and $\bar{D}(F) = 0$ and consequently, Equations (4.7) and (4.8) reduce to Equations (2.30) and (2.31). These equations also specialise to Equations (3.32) and (3.33) applicable to thin elastic shells of revolution by letting $A_1 = r_1$, $A_2 = r_0$, $\alpha_1 = \phi$ and $\alpha_2 = \theta$. It should be recalled

that within the frame work of the approximations involved in obtaining Equations (3.32) and (3.33), the two terms proportional to $\frac{\cos \phi}{r_0} = \frac{1}{r_0 r_1} \left(\frac{dr}{d\phi} \right)$ should be dropped, as mentioned in the last section.

A more accurate equation of equilibrium than Equation (4.8) can be shown to be:

$$\rho \ddot{W} + D(\nabla^4 W) = \left[\bar{D}(F) - N_{11} \kappa_{11} - N_{22} \kappa_{22} - 2N_{12} \kappa_{12} \right] \quad (4.10)$$

where, κ_{11} , κ_{22} and κ_{12} are given by Equation (4.4).

For the problem of buckling of the shell (either the linear buckling problem or the postbuckling problem), Equation (4.10) without the inertia term $\rho \ddot{W}$ is applicable provided the signs in front of N_{11} , N_{22} and N_{12} are changed from minus to plus so that for shell buckling, the governing equation becomes:

$$D(\nabla^4 W) + \bar{D}(F) - N_{11} \kappa_{11} - N_{22} \kappa_{22} - 2N_{12} \kappa_{12} = Z \quad (4.11)$$

where, $\bar{D}(F) = - \left[(N_{11}/r_1) + (N_{22}/r_2) \right]$ and Z represents the component of the applied surface load in the direction of the η_2 coordinate. For the linear buckling problem, the values of the stress resultants N_{11} , N_{22} and N_{12} just prior to buckling are determined from the linearised version of the

compatibility equation (4.7) wherein the term $\bar{D}(W)$ is dropped. The relationship between the stress function F and the stress resultants are defined by Equation (4.9). For the postbuckling problem, the stress resultants appearing in Equation (4.11) are determined from the compatibility equation (4.7) which is nonlinear.

With $ds_1 = A_1 d\alpha_1$ and $ds_2 = A_2 d\alpha_2$, Equations (4.7) and (4.8) can be written as:

$$\nabla^4 F = Eh \left\{ -\bar{D}(W) + \left(\frac{\partial^2 W}{\partial s_1 \partial s_2} \right)^2 - \left(\frac{\partial^2 W}{\partial s_1^2} \right) \left(\frac{\partial^2 W}{\partial s_2^2} \right) \right\} \quad (4.12)$$

and,

$$\rho \ddot{W} + D(\nabla^4 W) = \left\{ \bar{D}(F) + N_{11} \frac{\partial^2 W}{\partial s_1^2} + N_{22} \frac{\partial^2 W}{\partial s_2^2} + 2N_{12} \frac{\partial^2 W}{\partial s_1 \partial s_2} \right\} \quad (4.13)$$

In the above equations, $\nabla^4 = \nabla^2 \cdot \nabla^2$ and from equation (4.5),

$$\begin{aligned} \nabla^2(\dots) &= \left\{ \frac{1}{A_2} \frac{\partial}{\partial s_1} \left[A_2 \frac{\partial(\dots)}{\partial s_1} \right] + \frac{1}{A_1} \frac{\partial}{\partial s_2} \left[A_1 \frac{\partial(\dots)}{\partial s_2} \right] \right\} \\ &\approx \left[\frac{\partial^2(\dots)}{\partial s_1^2} + \frac{\partial^2(\dots)}{\partial s_2^2} \right] \end{aligned} \quad (4.14)$$

since terms involving products of the gradients of the Lamé parameters, namely, $\frac{\partial A_2}{\partial s_1}$ and $\frac{\partial A_1}{\partial s_2}$ are neglected in the approximation employed here.

With $N_{11} \approx -\frac{\partial^2 F}{\partial s_2^2}$, $N_{22} \approx -\frac{\partial^2 F}{\partial s_1^2}$ and $N_{12} \approx +\frac{\partial^2 F}{\partial s_1 \partial s_2}$

Equation (4.13) can be written as

$$\rho \ddot{W} + D(\nabla^4 W) = \left\{ \bar{D}(F) - \left(\frac{\partial^2 F}{\partial s_2^2} \right) \frac{\partial^2 W}{\partial s_1^2} - \left(\frac{\partial^2 F}{\partial s_1^2} \right) \frac{\partial^2 W}{\partial s_2^2} + 2 \left(\frac{\partial^2 F}{\partial s_1 \partial s_2} \right) \frac{\partial^2 W}{\partial s_1 \partial s_2} \right\}$$

We now define an operator (following Vol'mir)

$$O(W, F) \equiv (W_{s_1 s_1}) F_{s_2 s_2} + (W_{s_2 s_2}) F_{s_1 s_1} - 2 F_{s_1 s_2} W_{s_1 s_2}$$

so that we get,

$$\rho \ddot{W} + D(\nabla^4 W) = \left\{ \bar{D}(F) - O(W, F) \right\}$$

which is a neat, short way of writing Equation (4.13).

Since,

$$O(W, W) = 2 \left[(W_{s_1 s_1}) W_{s_2 s_2} - (W_{s_1 s_2})^2 \right]$$

we can write Equation (4.12) as

$$\nabla^4 F = Eh \left\{ -\bar{D}(W) - \frac{1}{2} O(W, W) \right\}$$

Equations (4.12) and (4.13) hold good for any thin elastic shell and it is readily verified that from these equations, Equations (2.34) and (2.35) applicable for thin flat plates and Equations (3.34) and (3.35) applicable for thin elastic shells of revolution can be obtained as special cases.

Section 5.

In this section, on the basis of an assumed vibration mode which is taken as the product of a function of time and function of the space variables, the modal equation is determined for the general case of thin elastic shells discussed in the last section. This modal equation can then be specialised for shells of revolution or flat plates or beams. The characteristic features of the modal equation as applicable to thin shells on the one hand and thin flat plates and straight beams on the other hand are discussed.

Let,

$$W(s_1, s_2, t) = B(t) \bar{W}(s_1, s_2) \quad (5.1)$$

where $B(t)$ is a generalised co-ordinate and $\bar{W}(s_1, s_2)$ is the mode shape assumed to satisfy all the boundary conditions (both geometric and natural). Introducing Equation (5.1) in the compatibility equation (4.12), we get

$$\nabla^4 F = Eh \left\{ -B(t) \bar{D}(\bar{W}) + B^2 \left[\left(\frac{\partial^2 \bar{W}}{\partial s_1 \partial s_2} \right)^2 - \left(\frac{\partial^2 \bar{W}}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_2^2} \right) \right] \right\} \quad (5.2)$$

where $[2]$,

$$\bar{D}(\bar{W}) = \left\{ \frac{1}{A_2} \frac{\partial}{\partial s_1} \left[\left(\frac{A_2}{r_2} \right) \frac{\partial \bar{W}}{\partial s_1} \right] + \frac{1}{A_1} \frac{\partial}{\partial s_2} \left[\left(\frac{A_1}{r_1} \right) \frac{\partial \bar{W}}{\partial s_2} \right] \right\} \quad (5.3)$$

Equation (5.2) is satisfied provided F is assumed as

$$F(s_1, s_2, t) = B(t) \bar{F}(s_1, s_2) + B^2 \bar{\bar{F}}(s_1, s_2) \quad (5.4)$$

So that,

$$\nabla^4 \bar{F} = -Eh [\bar{D}(\bar{W})] \quad (5.5)$$

and

$$\nabla^4 \bar{F} = Eh \left[\left(\frac{\partial^2 \bar{W}}{\partial s_1 \partial s_2} \right)^2 - \left(\frac{\partial^2 \bar{W}}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_2^2} \right) \right] \quad (5.6)$$

If the problem were linear so that Equation (4.12) is linear, then, $\bar{F} = 0$; that is, $F(s_1, s_2, t) = B(t) \bar{F}(s_1, s_2)$.

The equilibrium equation (4.13) can be written as

$$L(W, F) = 0 \quad (5.7)$$

where,

$$L(W, F) = \left\{ \rho \dot{W} + D(\nabla^4 W) - \bar{D}(F) - N_1 \frac{\partial^2 W}{\partial s_1^2} - N_{22} \frac{\partial^2 W}{\partial s_2^2} - 2N_{12} \frac{\partial^2 W}{\partial s_1 \partial s_2} \right\} \quad (5.8)$$

Using the Galerkin's method,

$$\int_S [L(W, F)] \delta W \, ds = 0 \quad (5.9)$$

where, $ds = A_1 A_2 d\alpha_1 d\alpha_2 = ds_1 ds_2$ and $\delta W = (\delta B) \bar{W}$

so that Equation (5.9) becomes:

$$\int_S [L(W, F)] \bar{W} \, ds = 0 \quad (5.10)$$

By means of Equations (5.1), (5.4) and (4.9a), Equation (5.8)

yields:

$$\begin{aligned}
 L(W, F) = & \left\{ (\rho \bar{W}) \ddot{B} + B \left[D \nabla^4 (\bar{W}) - \bar{D}(\bar{F}) \right] + B^2 \left[\left(\frac{\partial^2 \bar{F}}{\partial s_2^2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_1^2} \right) + \left(\frac{\partial^2 \bar{F}}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_2^2} \right) \right. \right. \\
 & \left. \left. - 2 \left(\frac{\partial^2 \bar{F}}{\partial s_1 \partial s_2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_1 \partial s_2} \right) - \bar{D}(\bar{F}) \right] + B^3 \left[\left(\frac{\partial^2 \bar{F}}{\partial s_2^2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_1^2} \right) + \left(\frac{\partial^2 \bar{F}}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_2^2} \right) \right. \right. \\
 & \left. \left. - 2 \left(\frac{\partial^2 \bar{F}}{\partial s_1 \partial s_2} \right) \left(\frac{\partial^2 \bar{W}}{\partial s_1 \partial s_2} \right) \right] \right\}
 \end{aligned} \tag{5.11}$$

It is helpful to define the following integrals:

$$I_1 = \rho \int_S (\bar{W})^2 ds.$$

$$I_2 = D \int_S (\nabla^4 \bar{W}) \bar{W} ds.$$

$$I_3 = \int_S [\bar{D}(\bar{F})] \bar{W} ds$$

$$I_4 = \int_S \left\{ -[\bar{D}(\bar{F})] + \frac{\partial^2 \bar{F}}{\partial s_2^2} \frac{\partial^2 \bar{W}}{\partial s_1^2} + \frac{\partial^2 \bar{F}}{\partial s_1^2} \frac{\partial^2 \bar{W}}{\partial s_2^2} - 2 \frac{\partial^2 \bar{F}}{\partial s_1 \partial s_2} \frac{\partial^2 \bar{W}}{\partial s_1 \partial s_2} \right\} \bar{W} ds$$

$$I_5 = \int_S \left\{ \frac{\partial^2 \bar{F}}{\partial s_2^2} \frac{\partial^2 \bar{W}}{\partial s_1^2} + \frac{\partial^2 \bar{F}}{\partial s_1^2} \frac{\partial^2 \bar{W}}{\partial s_2^2} - 2 \frac{\partial^2 \bar{F}}{\partial s_1 \partial s_2} \frac{\partial^2 \bar{W}}{\partial s_1 \partial s_2} \right\} \bar{W} ds \tag{5.12}$$

In terms of the above quantities, Equation (5.10) can be written as:

$$I_1 \ddot{B} + (I_2 - I_3) B + I_4 B^2 + I_5 B^3 = 0 \quad (5.13)$$

The solution to the above nonlinear ordinary differential equation involving $B(t)$ helps determine the amplitude versus frequency relation in the case of thin elastic shells. If the shell were undergoing small amplitude flexural vibration (that is, the linear problem), the associated frequency equation can be determined from the linearised version of the modal Equation (5.13) so that the governing linear differential equation becomes:-

$$I_1 \ddot{B} + (I_2 - I_3) B = 0 \quad (5.14)$$

Assuming solution of the form $B(t) = B_0 \cos \omega t$, the frequency ω as determined from Equation (5.14) is

$$\omega = \left[\frac{(I_2 - I_3)}{I_1} \right]^{\frac{1}{2}} \quad (5.15)$$

The modal equation (5.13) can be readily specialised for the case of shells of revolution for which $A_1 = r_1$, $A_2 = r_0$, $\alpha_1 = \phi$ and $\alpha_2 = \theta$ so that $ds_1 = A_1 d\alpha_1 = r_1 d\phi$ and $ds_2 = A_2 d\alpha_2 = r_0 d\theta$. Also, for a shell of revolution, the elemental area $ds = r_0 r_1 d\phi d\theta$.

It is of interest to obtain the modal equation for the case of thin elastic flat plates. This can be obtained in a manner analagous to that used to obtain Equation (5.13), and starting out with Equations (2.34) and (2.35). The required equation can also be obtained by specialising Equation (5.13) for the case of flat plates for which the radii of curvature of the median surface r_1 and r_2 are infinity. When $r_1 \rightarrow \infty$ and $r_2 \rightarrow \infty$, it is obvious from Equation (5.3) that,

$$\bar{D}(\bar{W}) = 0 \text{ and } \bar{D}(F) = 0 \quad (5.16)$$

Thus the compatibility Equation (2.34) is satisfied by

$$F(s_1, s_2, t) = B^2 \bar{F}(s_1, s_2) \quad (5.17)$$

comparing the above with Equation (5.4), it is seen that for the case of flat plates,

$$\bar{F}(s_1, s_2) = 0 \quad (5.18)$$

Because of Equations (5.16) and (5.18), it is clear from Equations (5.12) that for the case of flat plates,

$$I_3 = 0 ; I_4 = 0 \quad (5.19)$$

Consequently Equation (5.13) simplifies to

$$I_1 \ddot{B} + I_2 \dot{B} + I_5 B^3 = 0 \quad (5.20)$$

The above modal equation which is a nonlinear ordinary differential equation helps determine the amplitude versus frequency relationship for a thin elastic flat plate undergoing large amplitude flexural vibration. On comparing the last equation with Equation (5.13), it is seen that in the case of flat plates, the nonlinear term proportional to B^2 is absent. If the last equation is linearised, the governing equation for the small amplitude flexural vibration of a thin elastic flat plate is obtained, from which, the frequency ω (rad/second) can be determined as

$$\omega = \sqrt{I_2 / I_1} \quad (5.21)$$

The modal equation in the case of straight beams undergoing large amplitude flexural vibration can also be shown to be given by Equation (5.20). For this, it is easier to work with the governing equation for the large amplitude flexural vibration of straight beams, namely,

$$\rho \ddot{W} + EI \frac{\partial^4 W}{\partial x^4} - N_{11} \frac{\partial^2 W}{\partial x^2} = 0 \quad (5.22)$$

where, $N_{11} = Ea C_1$ and 'a' is the beam cross sectional area.

From the equilibrium equation in the x-coordinate direction, it can be shown that $\frac{dC_1}{dx} = 0$ so that $C_1 = C_0 = \text{Constant}$. Thus, $N_{11} = Ea C_0$ where $C_1 = \left[u_{,x} + \frac{1}{2}(w_{,x})^2 \right]$. It can be

shown that $C_0 = \frac{1}{\ell} \int_0^\ell (u_{,x} + \frac{1}{2} w_{,x}^2) dx$

$$= \frac{1}{\ell} \left[(u)_0^\ell + \frac{1}{2} \int_0^\ell (w_{,x})^2 dx \right].$$

If $(u)_0^\ell = 0$, $C_0 = \frac{1}{2\ell} \int_0^\ell (w_{,x})^2 dx$ so that $N_{II} = \frac{E\alpha}{2\ell} \int_0^\ell (w_{,x})^2 dx$.

Assuming $W(x,t) = \Lambda(t) \cdot \bar{W}(x)$ and using the Galerkin's method as before, the modal equation can be shown to be

$$J_1 \ddot{A} + J_2 \dot{A} + J_3 A^3 = 0 \quad (5.23)$$

where,

$$J_1 = \rho \int_0^\ell (\bar{W})^2 dx ; \quad J_2 = EI \int_0^\ell \left(\frac{d^4 \bar{W}}{dx^4} \right) \bar{W} dx$$

$$J_3 = -\frac{E\alpha}{2} \int_0^\ell \left(\frac{d\bar{W}}{dx} \right)^2 \left(\frac{d^2 \bar{W}}{dx^2} \right) \bar{W} dx. \quad (5.24)$$

It is of interest to note that the modal equation [see Equation (5.13) or (5.20)] can also be obtained by applying the Lagrange's equation of motion, namely,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{B}} \right) - \frac{\partial}{\partial B} (T - U) = 0 \quad (5.25)$$

where, T is the kinetic energy of the system, U represents the strain energy, B is the generalised coordinate and therefore, \dot{B} is the generalised velocity. With,

$$T = \frac{\rho}{2} \int_s (\dot{W})^2 ds$$

where W is given by Equation (5.1), we get

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{B}} \right) = I_1 \ddot{B}$$

In the above, I_1 is given by Equation (5.12) so that the above represents the first term of Equation (5.13). Similarly the other terms of Equation (5.13) or (5.20) can be shown to stem from the second term of Lagrange's equation given by Equation (5.25).

Thus, the modal equation from which the frequency versus amplitude relationship is determined in the case of the large amplitude flexural vibration of straight beams and plates is of the same type [see Equation (5.20)] whereas, in the case of thin elastic shells the governing equation [see Equation (5.13)] has an additional term proportional to B^2 . It is worth investigating whether this circumstance has anything to do with the fact that in the existing literature, most of the available solutions for the case of thin shells are found to give rise to nonlinearity of the "softening type" (that is, frequency decreases with increasing amplitude) whereas in the case of flat plates of different geometries the nonlinearity is always found to be of the "hardening type" (that is, frequency increases with increasing amplitude). In fact, based on the survey paper [6] and [7], it is possible to draw the following general conclusions regarding the type of nonlinearity exhibited by the different elastic bodies.

(A) Straight beams - - - - - Hardening type

(B) Plates { (1) Rectangular
(2) Circular
(3) Elliptical
(4) Triangular
(5) Skew
(6) Rectangular Sandwich type } ----- Hardening type

(C) Rings - - - - - Softening type

(D) Shells* { (1) Circular cylindrical
(2) Oval cylindrical
(3) Spherical } -- Softening type

The above summary of results raises the question as to why in the case of straight beams and flat plates (irrespective of plate geometry), the nonlinearity is always of the hardening type whereas in the case of bodies like rings and shells the undeformed median surface of which have curvature to begin with, the nonlinearity is of the softening type.

Equations (5.13) and (5.20) can be written formally as

$$\frac{d^2\theta}{dt^2} + a_0\theta + b\theta^2 + c\theta^3 = 0 \quad (5.26)$$

where, a_0, b and c are real numbers and $\theta = \theta(t)$. It will

* In the case of cylindrical shells, it is assumed that the ratio of the axial to circumferential half-waves of the vibration mode is small. If this ratio is large, the nonlinearity is found to be of the hardening type.

be easily recognised that the above equation also represents the dynamic equation of a mass-spring combination where the spring restoring force is nonlinear and is defined by a cubic expression of the form $(a_0^2 \theta + b\theta^2 + c\theta^3)$.

Multiply each term of Equation (5.26) by $(\frac{d\theta}{dt} dt)$ and integrate to obtain

$$(\dot{\theta})^2 = C_0 - \left[a_0^2 \theta^2 + \frac{2}{3} b \theta^3 + \frac{c}{2} \theta^4 \right]$$

where C_0 is a constant of integration to be determined from initial conditions on θ and $\dot{\theta}$. From the last equation, we can finally obtain

$$t = \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{Q(\theta)}} \quad (5.27)$$

where $Q(\theta) = \left[C_0 - \left(a_0^2 \theta^2 + \frac{2}{3} b \theta^3 + \frac{c}{2} \theta^4 \right) \right]$ is a quartic in θ and $\theta_0 = \theta(t = 0)$. The integral appearing in Equation (5.27) is an Elliptic integral of the first kind. The nature of θ versus t curve will depend upon the constants a_0^2 , b , c and C_0 . For the flat plate problem the governing equation of which is Equation (5.20), the coefficient $b = 0$ in Equation (5.26) so that $Q(\theta)$ appearing in Equation (5.27) becomes a biquadratic. This is true also of the problem involving straight beams [see Equation (5.23)].

When $b = 0$ in Equation (5.26), the solution to the resulting differential equation where $c > 0$ can be shown

to be (see page 26, Ref.8)

$$t = \frac{1}{(a_0^2 + c\theta_0^2)^{1/2}} F(\lambda, \varphi) \quad (5.28)$$

where,

$$F(\lambda, \varphi) = \int_0^\varphi \frac{d\psi}{\sqrt{1 - \lambda^2 \sin^2 \psi}} \quad (5.29)$$

is the elliptic integral of the first kind and

$$\lambda^2 = \frac{\alpha}{2(1+\alpha)} \quad \text{where } \alpha = \frac{c\theta_0^2}{a_0^2} \quad (5.30)$$

obviously $\lambda^2 < 1$. The initial conditions are assumed to be $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$. It can also be shown [8] that the period of oscillation of the dynamical system for which the differential equation (5.26) with $b = 0$ applies is given by

$$\tau_0 = \frac{4F(\lambda, \frac{\pi}{2})}{a_0\sqrt{1+\alpha}} \quad (5.31)$$

where, $F(\lambda, \frac{\pi}{2})$ is the complete elliptic integral of the first kind with modulus λ . Furthermore, it is known that the period τ decreases with increase in a_0^2 , c , θ_0 either individually or collectively [8]. When $c = 0$, we have the linear problem so that

$$(\tau)_{\text{linear}} \equiv \tau^* = \frac{2\pi}{a_0}$$

Thus,

$$\left(\frac{\tau}{\tau^*}\right) = \frac{2}{\pi} \frac{F(\lambda, \frac{\pi}{2})}{\sqrt{1+\alpha}} \quad (5.32)$$

The above ratio which is equal to unity for $c = 0$ can be shown to be less than unity for all $c > 0$ [8].

Consequently, the associated non-dimensional frequency

$(\omega/\omega^*) = (\tau^*/\tau)$ where ω is the frequency for the nonlinear problem and ω^* is that for the linear problem, increases.

Thus the nonlinearity is of the hardening type. For any fixed set of values of a_0^2 and c , since the period τ decreases with increasing amplitude θ_0 , the nondimensional frequency (ω/ω^*) will increase with increase in the amplitude θ_0 which is exactly the result mentioned earlier for the case of flat plates. Thus we arrive at the important conclusion that in the case of large amplitude flexural vibration of thin flat plates and straight beams for which the modal equation has been shown to be given by Equation (5.26) wherein $b = 0$ and provided $c > 0$, the frequency increases with increasing amplitude (hardening type of nonlinearity). Note that this conclusion is of a very general nature and is not restricted by any particular plate geometry.

A conclusion of a similar broad nature in the case of thin shells would be most interesting, if such can be found. To this end, we examine the modal equation (5.13) applicable for any thin shell where the assumed mode shape is given by Equation (5.1). Equation (5.13) is of the general form of Equation (5.26) and by comparing these two equations, we see that

$$a_0^2 = (I_2 - I_3) \quad (5.33)$$

where I_2 and I_3 are both positive and it is assumed $I_2 > I_3$ which can be physically justified from the frequency for the

linear problem as given by Equation (5.15). The coefficient I_3 which depends on the quantity $\bar{D}(\bar{F})$ vanishes for the case of flat plates as can be seen from Equation (5.3). Thus the value of the coefficient a_0^2 for a thin elastic shell is less than that for the case of the corresponding flat plate obtained by putting the principal radii of curvature r_1 and r_2 each tending to infinity. If a_0^2 decreases, Equation (5.26) with $b = 0$ will give rise to a large value for period τ which means smaller value for the frequency. Thus the trend of the non-linearity is towards the softening type which agrees with the result of almost all the investigations so far on thin elastic shells of various geometries. But this result based on Equation (5.26) with $b = 0$ is not quite conclusive because in the case of thin shells, the modal equation given by Equation (5.26) has the term $b\theta^2$ in it. With $b \neq 0$, a general solution to Equation (5.26) seems difficult. However, let us examine the equation

$$\frac{d^2\theta}{dt^2} + a_0^2\theta + b\theta^2 = 0 \quad (5.34)$$

where, a_0^2 and b are both positive numbers and the initial conditions are $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$. Assuming a solution of the form

$$\theta = A_0 + A_1 \cos \omega t + A_3 \cos 3\omega t$$

it can be shown (see page 51 of Ref.8) that

$$\left(\frac{\omega}{\omega_0}\right)^2 \approx \left[1 - \frac{5}{12} \left(\frac{b^2}{a_0^4}\right) \theta_0^2\right] \quad (5.35)$$

where ω_0 is the frequency of the nonlinear problem and

$\omega^* = a_0$ is the frequency of the linear problem. From equation (5.35) it is readily seen that the nondimensional frequency decreases with increasing amplitude θ_0 which again implies softening type of nonlinearity. Thus in the case of thin shells, the decrease in the value of the coefficient a_0^2 due to I_3 [see Equation (5.33)] and the presence of the term $b\theta^2$ in equation (5.26) each contributes individually to the softening type of nonlinearity.

So far, we have worked with an assumed one term mode shape given by Equation (5.1) and determined the modal equation for the generalised co-ordinate $B(t)$. The nonlinearity of the problem makes the analysis based on an assumed mode shape consisting of a finite number of terms very difficult. Quite naturally, the analysis becomes more complicated as the number of terms in the assumed mode shape increases. All the same, it is of interest to obtain and examine the nature of the modal equations that result when a series is assumed for the mode shape. Therefore, let

$$W(s_1, s_2, t) = \sum_i B_i(t) \bar{W}_i(s_1, s_2) \quad (5.36)$$

where, B_i 's are the generalised coordinates and \bar{W}_i 's which are functions of the space variables s_1 and s_2 are assumed to satisfy all the boundary conditions (both geometric and natural). On introducing Equation (5.36) into the compatibility equation (4.12) and examining the resulting equation,

it will be seen that the stress function F has to be of the form,

$$F(s_1, s_2, t) = \left\{ \sum_i \left[B_i \bar{F}_i(s_1, s_2) + (B_i)^2 \bar{\bar{F}}_i \right] + \sum_{i \neq j} \sum_j B_i B_j \bar{F}_i^* \bar{F}_j^* \right\} \quad (5.37)$$

where, the three sets of functions $\{\bar{F}_i\}$, $\{\bar{\bar{F}}_i\}$ and $\{\bar{F}_i^*\}$

which are functions of only s_1 and s_2 are determined from the following equations:-

$$\nabla^4(\bar{F}_i) = -Eh \bar{D}(\bar{w}_i) \quad (5.38)$$

$$\nabla^4(\bar{\bar{F}}_i) = Eh \left[(\bar{w}_{i, s_1 s_2})^2 - (\bar{w}_{i, s_1 s_1})(\bar{w}_{i, s_2 s_2}) \right] \quad (5.39)$$

$$\nabla^4(\bar{F}_i^* \bar{F}_j^*) = 2 Eh \left[(\bar{w}_{i, s_1 s_2})(\bar{w}_{j, s_1 s_2}) - (\bar{w}_{i, s_1 s_1})(\bar{w}_{j, s_2 s_2}) \right] \quad (5.40)$$

Because of Equation (5.36),

$$\delta W = (\delta B_p) \bar{w}_p \quad (5.41)$$

where, p takes on (one at a time) each one of the values that i can assume. Thus, Equation (5.9) yields

$$\int_S [L(W, F)] \bar{w}_p ds = 0 \quad (5.42)$$

where, $L(W, F)$ is given by Equation (5.8). If W as given by Equation (5.36) and F as given by Equation (5.37) are introduced into Equation (5.8) and recalling the definition

of stress resultants in terms of F given by Equation (4.9a) Equation (5.42) can be used to yield the following modal equation:

$$\begin{aligned} & \left\{ (I_1)_{ip} (\ddot{B}_i) + [(I_2)_{ip} - (I_3)_{ip}] B_i + (I_4)_{ip} (B_i)^2 \right. \\ & + (I_5)_{ip} (B_i)^3 + \sum_j [(I_6)_{ijp} (B_i B_j) + (I_7)_{ijp} (B_i^2 B_j)] \\ & \left. + \sum_j \sum_k (I_8)_{ijkp} (B_i B_j B_k) \right\} = 0 \end{aligned} \quad (5.43)$$

In the above, p takes on (one at a time) each one of the values that i takes and $j \neq k \neq i$. The coefficients $(I_1)_{ip}$, $(I_2)_{ip}$ etc. are surface integrals, the integrands of which involve functions of \bar{W}_i 's, \bar{F}_i 's, \bar{F}_i^* 's and $F_i^* F_j^*$. The explicit nature of the coefficients are not given here, as they are complicated expressions. For each value that p can take, we can write an equation using Equation (5.43) so that there are as many equations as there are p 's (that is, i 's). This set of nonlinear equations are coupled in the generalised coordinates B_i 's. If a one term solution is considered so that $i = 1$ (and since $j \neq k \neq i$ so that the quantities with j and k as subscripts do not exist), Equation (5.43) will reduce to Equation (5.13). From Equation (5.43) it is readily seen that the nonlinear terms are of two types, namely, of the quadratic type [B_i^2 and $B_i B_j$ terms] and of the cubic type (B_i^3 , $B_i^2 B_j$ and $B_i B_j B_k$). If in Equation (5.43),

the terms involving coupling between the generalised coordinates are neglected (that is, terms involving $B_i B_j$, $B_i^2 B_j$ and $B_i B_j B_k$), then, the resulting equation has the form of Equation (5.13).

If the above analysis starting with Equation (5.36) is carried out for the case of flat plates, the modal equation analagous to Equation (5.43) which is valid for thin elastic shells can be shown to be:-

$$\left\{ (I_1)_{ip} \ddot{B}_i + (I_2)_{ip} \dot{B}_i + (I_5)_{ip} B_i^3 + \sum_j (I_7)_{ijp} B_i^2 B_j + \sum_j \sum_k (I_8)_{ijkp} (B_i B_j B_k) \right\} = 0 \quad (5.44)$$

where, p as before takes on (one at a time) each one of the values that i takes and $j \neq k \neq i$. Equation (5.44) represents as many equations as there are p 's or i 's. Comparing Equation (5.44) with Equation (5.43), it is seen that in the case of flat plates, the nonlinear terms of the quadratic type (that is, terms proportional to B_i^2 and $B_i B_j$) are absent and only nonlinear terms of the cubic type (that is, B_i^3 , $B_i^2 B_j$ and $B_i B_j B_k$) are present. A similar observation was made earlier with reference to Equations (5.13) and (5.20).

A different approach to obtain the modal equations is possible. This involves writing the expressions for the kinetic energy T and the strain energy U of the elastic body. The expression for U can be obtained from Equations (2.6), (2.7)

and (2.9) wherein the appropriate relations for the median surface strains ϵ_1 , ϵ_2 and ω and for the curvatures κ_{11} , κ_{22} and κ_{12} are used. Thus V and U can be obtained in terms of the displacement components u , v and w . Assuming for u , v and w the product type of functions involving the generalised coordinates [see for example, Equation (5.1)] and using the Lagrange's equation (5.25), the modal equations involving the generalised coordinates can be determined. In this approach, the compatibility equation (4.12) and the simplifications and approximations embodied in it or in the equilibrium equation as given by Equation (4.13) are not involved. However, in this approach, a set of coupled non-linear ordinary differential equations involving the generalised coordinates will have to be solved. This approach of deriving the modal equations is similar to what Mayers and Wrenn [9] employed in the solution to the problem of non-linear flexural vibration of thin circular cylindrical shells.

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Section 6.

In this section the problem of buckling (linear problem) and postbuckling of plates and shells are discussed from a general point of view. In Section 4, the governing equations for the postbuckling problem of thin elastic shells have been obtained and they are the equilibrium equation (4.11) and the compatibility equation (4.7) or (4.12). It is helpful to reproduce these equations here.

$$D(\nabla^4 W) - \left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) - N_{11} \kappa_{11} - N_{22} \kappa_{22} - 2N_{12} \kappa_{12} = Z \quad (6.1)$$

and,

$$\nabla^4 F = Eh \left\{ -\bar{D}(W) + \left(\frac{\partial^2 W}{\partial s_1 \partial s_2} \right)^2 - \left(\frac{\partial^2 W}{\partial s_1^2} \right) \left(\frac{\partial^2 W}{\partial s_2^2} \right) \right\} \quad (6.2)$$

In Equation (6.1), $-\left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) = \bar{D}(F)$

For the linear or classical buckling problem, the values of the stress resultants N_{11} , N_{22} and N_{12} just prior to buckling do not depend upon the buckling mode W so that they are determined from the linearised version of the compatibility equation (6.2) wherein the term $\bar{D}(W)$ which depends upon the buckling mode W , is neglected. Thus the stress resultants for the linear buckling problem are determined from the compatibility equation $\nabla^4 F = 0$. For the post-buckling problem, the stress resultants appearing in Equation (6.1) are determined from the compatibility equation (6.2) which is nonlinear.

The governing equations for flat plates can be obtained from Equations (6.1) and (6.2) by recalling that for flat plates, the principal radii of curvatures r_1 and r_2 are both infinity so that the operator \bar{D} vanishes, as can be seen from Equation (4.6).

As in the last section, the function W is assumed as a series so that

$$W(s_1, s_2) = \sum_i B_i \bar{W}_i(s_1, s_2) \quad (6.3)$$

where B_i 's are coefficients to be determined and \bar{W}_i 's are functions each of which is assumed to satisfy all the boundary conditions (both geometric and natural) of the problem. With the assumed mode shape given by Equation (6.3), the compatibility equation (6.2) is satisfied if $F(s_1, s_2)$ is assumed as:

$$F(s_1, s_2) = \left\{ p^* F_0(s_1, s_2) + \sum_i [B_i \bar{F}_i(s_1, s_2) + B_i^2 \bar{F}_i] + \sum_i \sum_{j \neq i} B_i B_j F_i^* F_j^* \right\} \quad (6.4)$$

where, p^* is the applied load (other than any lateral load Z) that gives rise to the buckling problem. When $p^* = 0$, Equation (6.4) reduces to Equation (5.29). The set of functions $\{\bar{F}_i\}$, $\{\bar{F}_i^*\}$ and $\{F_j^*\}$ are determined from Equations (5.38), (5.39) and (5.40). The function F_0 appearing in Equation (6.4) is determined from the biharmonic equation

$$\nabla^4 F_0 = 0 \quad (6.5)$$

For the linear buckling problem, it has been mentioned that the compatibility equation is $\nabla^4 F_0 = 0$ so that, Equation (6.4) simplifies to

$$F_{\text{linear}} = p^* F_0(s_1, s_2) \quad (6.6)$$

where, p^* is the critical value of the load at which buckling occurs so that p^* is the quantity to be determined in the linear buckling problem.

The stress resultants N_{11} , N_{22} and N_{12} are defined in terms of the stress function F by Equations (4.9) from which we write (in view of $ds_1 = A_1 d\alpha_1$ and $ds_2 = A_2 d\alpha_2$):

$$\begin{aligned} N_{11} &= - \left[\frac{\partial^2 F}{\partial s_2^2} + \frac{1}{A_2} \left(\frac{\partial A_2}{\partial s_1} \right) \left(\frac{\partial F}{\partial s_1} \right) \right] \\ N_{22} &= - \left[\frac{\partial^2 F}{\partial s_1^2} + \frac{1}{A_1} \left(\frac{\partial A_1}{\partial s_2} \right) \left(\frac{\partial F}{\partial s_2} \right) \right] \\ N_{12} &= \left[\frac{\partial^2 F}{\partial s_1 \partial s_2} - \frac{1}{A_1} \left(\frac{\partial A_1}{\partial s_2} \right) \frac{\partial F}{\partial s_1} - \frac{1}{A_2} \left(\frac{\partial A_2}{\partial s_1} \right) \frac{\partial F}{\partial s_2} \right] \end{aligned} \quad (6.7)$$

The curvature expressions given by Equations (4.4) can be written as:

$$\begin{aligned} \kappa_{11} &= - \left[\frac{\partial^2 w}{\partial s_1^2} + \frac{1}{A_1} \left(\frac{\partial A_1}{\partial s_2} \right) \left(\frac{\partial w}{\partial s_2} \right) \right] \\ \kappa_{22} &= - \left[\frac{\partial^2 w}{\partial s_2^2} + \frac{1}{A_2} \left(\frac{\partial A_2}{\partial s_1} \right) \left(\frac{\partial w}{\partial s_1} \right) \right] \\ \kappa_{12} &= \left[- \frac{\partial^2 w}{\partial s_1 \partial s_2} + \left(\frac{\partial A_1}{\partial s_2} \right) \left(\frac{1}{A_1} \right) \frac{\partial w}{\partial s_1} + \left(\frac{\partial A_2}{\partial s_1} \right) \left(\frac{1}{A_2} \right) \frac{\partial w}{\partial s_2} \right] \end{aligned} \quad (6.8)$$

If the same approximation as used in the previous sections, namely, neglecting terms involving products of the gradients of the Lamé' parameters such as $(\frac{\partial A_1}{\partial s_2})$ and $(\frac{\partial A_2}{\partial s_1})$, is made use of, the stress resultants and curvatures as given above can be simplified to:

$$N_{11} \approx -\left(\frac{\partial^2 F}{\partial s_2^2}\right); \quad N_{22} \approx -\left(\frac{\partial^2 F}{\partial s_1^2}\right); \quad N_{12} \approx +\left(\frac{\partial^2 F}{\partial s_1 \partial s_2}\right) \quad (6.9)$$

and,

$$\kappa_{11} \approx -\left(\frac{\partial^2 W}{\partial s_2^2}\right); \quad \kappa_{22} \approx -\left(\frac{\partial^2 W}{\partial s_1^2}\right); \quad \kappa_{12} \approx -\left(\frac{\partial^2 W}{\partial s_1 \partial s_2}\right) \quad (6.10)$$

Using the above relations, the equilibrium equation (6.1) can be written as:

$$\begin{aligned} L(W, F) \equiv D(\nabla^4 W) + \left[\frac{1}{r_1} \frac{\partial^2 F}{\partial s_2^2} + \frac{1}{r_2} \frac{\partial^2 F}{\partial s_1^2}\right] - \frac{\partial^2 F}{\partial s_2^2} \left(\frac{\partial^2 W}{\partial s_1^2}\right) - \frac{\partial^2 F}{\partial s_1^2} \left(\frac{\partial^2 W}{\partial s_2^2}\right) \\ + 2 \frac{\partial^2 F}{\partial s_1 \partial s_2} \left(\frac{\partial^2 W}{\partial s_1 \partial s_2}\right) - Z = 0 \end{aligned} \quad (6.11)$$

Employing, as in the previous sections, the Galerkin's method, we get,

$$\int_A [L(W, F)] \delta W \, dA = 0$$

where, $dA = ds_1 \, ds_2$ is the elemental area on the shell median surface.

With $\delta W = (\delta B_p) \bar{W}_p$, the last equation yields

$$\int_A [L(W, F)] \bar{W}_p \, dA = 0 \quad (6.12)$$

By means of Equation (6.3), we can write from Equation (6.11)

$$L(W, F) = \sum_i \left\{ I_0 + (I_1)_i B_i + (I_2)_i B_i^2 + (I_3)_i B_i^3 + \sum_j (I_4)_{ij} B_i B_j + \sum_j (I_5)_{ij} B_i^2 B_j \right. \\ \left. + \sum_j \sum_k (I_6)_{ijk} B_i B_j B_k \right\} = 0 \quad (6.13)$$

where $j \neq k \neq i$ and $I_0, (I_1)_i$ etc. are defined as follows:

$$I_0 = \left[-Z + p^* \left(\frac{1}{r_1} \frac{\partial^2 F_0}{\partial s_2^2} + \frac{1}{r_2} \frac{\partial^2 F_0}{\partial s_1^2} \right) \right]$$

$$(I_1)_i = \left\{ D(\nabla^4 \bar{w}_i) + \frac{1}{r_1} \frac{\partial^2 \bar{F}_i}{\partial s_2^2} + \frac{1}{r_2} \frac{\partial^2 \bar{F}_i}{\partial s_1^2} - p^* \left[\left(\frac{\partial^2 \bar{w}_i}{\partial s_1^2} \right) \left(\frac{\partial^2 F_0}{\partial s_2^2} \right) + \left(\frac{\partial^2 \bar{w}_i}{\partial s_2^2} \right) \left(\frac{\partial^2 F_0}{\partial s_1^2} \right) - 2 \left(\frac{\partial^2 \bar{w}_i}{\partial s_1 \partial s_2} \right) \left(\frac{\partial^2 F_0}{\partial s_1 \partial s_2} \right) \right] \right\}$$

$$(I_2)_i = \left[\frac{1}{r_1} \frac{\partial^2 \bar{F}_i}{\partial s_2^2} + \frac{1}{r_2} \frac{\partial^2 \bar{F}_i}{\partial s_1^2} - \left(\frac{\partial^2 \bar{F}_i}{\partial s_2^2} \right) \left(\frac{\partial^2 \bar{w}_i}{\partial s_1^2} \right) - \left(\frac{\partial^2 \bar{F}_i}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{w}_i}{\partial s_2^2} \right) + 2 \left(\frac{\partial^2 \bar{F}_i}{\partial s_1 \partial s_2} \right) \left(\frac{\partial^2 \bar{w}_i}{\partial s_1 \partial s_2} \right) \right]$$

$$(I_3)_i = - \left[\left(\frac{\partial^2 \bar{F}_i}{\partial s_2^2} \right) \left(\frac{\partial^2 \bar{w}_i}{\partial s_1^2} \right) + \left(\frac{\partial^2 \bar{F}_i}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{w}_i}{\partial s_2^2} \right) - 2 \left(\frac{\partial^2 \bar{F}_i}{\partial s_1 \partial s_2} \right) \left(\frac{\partial^2 \bar{w}_i}{\partial s_1 \partial s_2} \right) \right]$$

$$(I_4)_{ij} = \left[\frac{1}{r_1} \frac{\partial^2}{\partial s_2^2} (F_i^* F_j^*) + \frac{1}{r_2} \frac{\partial^2}{\partial s_1^2} (F_i^* F_j^*) - \left(\frac{\partial^2 \bar{F}_i}{\partial s_2^2} \right) \left(\frac{\partial^2 \bar{w}_j}{\partial s_1^2} \right) - \left(\frac{\partial^2 \bar{F}_i}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{w}_j}{\partial s_2^2} \right) + 2 \left(\frac{\partial^2 \bar{F}_i}{\partial s_1 \partial s_2} \right) \left(\frac{\partial^2 \bar{w}_j}{\partial s_1 \partial s_2} \right) \right]$$

$$(I_5)_{ij} = - \left[\left(\frac{\partial^2 \bar{F}_i}{\partial s_2^2} \right) \left(\frac{\partial^2 \bar{w}_j}{\partial s_1^2} \right) + \left(\frac{\partial^2 \bar{F}_i}{\partial s_1^2} \right) \left(\frac{\partial^2 \bar{w}_j}{\partial s_2^2} \right) - 2 \left(\frac{\partial^2 \bar{F}_i}{\partial s_1 \partial s_2} \right) \left(\frac{\partial^2 \bar{w}_j}{\partial s_1 \partial s_2} \right) + \frac{\partial^2}{\partial s_2^2} (F_i^* F_j^*) \left(\frac{\partial^2 \bar{w}_i}{\partial s_1^2} \right) \right. \\ \left. + \frac{\partial^2}{\partial s_1^2} (F_i^* F_j^*) \left(\frac{\partial^2 \bar{w}_i}{\partial s_2^2} \right) - 2 \frac{\partial^2}{\partial s_1 \partial s_2} (F_i^* F_j^*) \left(\frac{\partial^2 \bar{w}_i}{\partial s_1 \partial s_2} \right) \right]$$

and,

$$(I_6)_{ijk} = - \left[\frac{\partial^2}{\partial s_2^2} (F_i^* F_j^*) \frac{\partial^2 \bar{w}_k}{\partial s_1^2} + \frac{\partial^2}{\partial s_1^2} (F_i^* F_j^*) \frac{\partial^2 \bar{w}_k}{\partial s_2^2} - 2 \frac{\partial^2}{\partial s_1 \partial s_2} (F_i^* F_j^*) \frac{\partial^2 \bar{w}_k}{\partial s_1 \partial s_2} \right]. \quad (i. 14)$$

Define,

$$J_{op} = \int_A I_o \bar{w}_p dA$$

$$(J_1)_{ip} = \int_A (I_1)_i \bar{w}_p dA$$

$$(J_2)_{ip} = \int_A (I_2)_i \bar{w}_p dA.$$

$$(J_3)_{ip} = \int_A (I_3)_i \bar{w}_p dA$$

$$(J_4)_{ijp} = \int_A (I_4)_{ij} \bar{w}_p dA$$

$$(J_5)_{ijp} = \int_A (I_5)_{ij} \bar{w}_p dA$$

$$(J_6)_{ijkp} = \int_A (I_6)_{ijk} \bar{w}_p dA$$

(6.15)

Note that $j \neq k \neq i$

Using Equations (6.13), (6.14) and (6.15) in Equation (6.12), we finally get:

$$J_{op} + \sum_i \left[(J_1)_{ip} B_i + (J_2)_{ip} B_i^2 + (J_3)_{ip} B_i^3 + \sum_j (J_4)_{ijp} B_i B_j + \sum_j (J_5)_{ijp} B_i^2 B_j + \sum_j \sum_k (J_6)_{ijkp} (B_i B_j B_k) \right] = 0 \quad (6.16)$$

Since p can take on (one at a time) each one of the values i can take on, Equation (6.16) represents as many equations as there are p 's; that is, i 's. This set of algebraic equations is nonlinear because each equation has cubic terms (B_i^3 , $B_i^2 B_j$ and $B_i B_j B_k$ terms) and also quadratic terms (B_i^2 and $B_i B_j$ terms). Furthermore, each equation has a right hand side given by J_{op} . J_{op} vanishes for a flat plate ($r_1 \rightarrow \infty$ and $r_2 \rightarrow \infty$) provided the lateral loading function $Z = 0$; in such a case, the set of algebraic equations has no constant term. It is to be noted that when $J_{op} = 0$, Equation (6.16) is similar in form to Equation (5.35) wherein the inertia term \ddot{B}_i is dropped and B_i is treated as a parameter independent of time. For the postbuckling problem of the shell, we have to solve simultaneously a set of cubic equations in the coefficients B_i 's given by Equation (6.16). These cubic equations have a constant term even when the lateral load $Z = 0$, as can be seen from the definition of J_{op} given by Equation (6.15).

For the linear buckling problem of the shell where the buckling loads p^* is to be determined, Equation (6.16) is linearised to give

$$\sum_i (J_i)_{ip} B_i + J_{op} = 0 \quad (6.17)$$

The above represents a set of linear algebraic equations in B_i 's obtained by writing an equation for each value of p .

For determining the buckling loads p^* , the characteristic equation is determined from the condition that the coefficients B_i of the assumed buckling mode shape become arbitrarily large. Thus, the condition becomes:

$$\Delta \equiv \left| \sum_i (J_i)_{ip} \right| = 0 \quad (6.18)$$

which on expansion gives the following determinantal equation:

$$\begin{vmatrix} (J_1)_{11} & (J_1)_{21} & (J_1)_{31} & (J_1)_{41} & \dots & (J_1)_{N1} \\ (J_1)_{12} & (J_1)_{22} & (J_1)_{32} & (J_1)_{42} & \dots & (J_1)_{N2} \\ (J_1)_{13} & (J_1)_{23} & (J_1)_{33} & (J_1)_{43} & \dots & (J_1)_{N3} \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0 \quad (6.19)$$

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It is of interest to specialise Equation (6.16) for the case of flat plates for which $r_1 \rightarrow \infty$ and $r_2 \rightarrow \infty$.

Thus if $Z = 0$ so that $J_{op} = 0$, the governing algebraic equation for the postbuckling behavior of flat plates is obtained from Equation (6.16) as:

$$\begin{aligned} \sum_i B_i \left[(J_1)_{ip} + (J_2)_{ip} B_i + (J_3)_{ip} B_i^2 + \sum_j (J_4)_{ijp} B_j \right. \\ \left. + \sum_j (J_5)_{ijp} B_i B_j + \sum_j \sum_k (J_6)_{ijkp} (B_j B_k) \right] = 0 \end{aligned} \quad (6.20)$$

Since in general, $B_i \neq 0$, the above reduces to a quadratic equation, viz.,

$$\begin{aligned} \sum_i \left\{ (J_3)_{ip} B_i^2 + \sum_j (J_5)_{ijp} (B_i B_j) + \sum_j \sum_k (J_6)_{ijkp} (B_j B_k) \right. \\ \left. + (J_2)_{ip} B_i + \sum_j (J_4)_{ijp} B_j + (J_1)_{ip} \right\} = 0 \end{aligned} \quad (6.21)$$

Thus, we conclude that for the case of postbuckling behavior of flat plates, the set of algebraic equations in the coefficients B_i 's to be solved simultaneously consists of quadratic equations in B_i 's whereas for the case of the postbuckling behavior of thin shells, the set of algebraic equations to be solved simultaneously consists of cubic equations in B_i 's and this situation is true when $Z = 0$.

It has been mentioned that for the linear buckling problem, the stress function determined from the compatibility equation $\nabla^4 F_0 = 0$ is given by Equation (6.6). For the postbuckling problem, the stress function F is determined from the compatibility equation (6.2) which is nonlinear. Based

on this nonlinear compatibility equation and the assumed postbuckling mode given by Equation (6.3), the set of coupled nonlinear algebraic equations in the coefficients B_i 's are given by Equation (6.16). This set consists of cubic equations in B_i 's. An approximation to this set of equations can be obtained by working with the linearised version of Equation (6.2), namely,

$$\nabla^4 F = -Eh \bar{D}(w) \quad (6.22)$$

The last equation involves the simplest form of coupling between w and F that is possible for the compatibility equation. The last equation together with Equation (6.3) can be shown to lead to the stress function F given by

$$F(s_1, s_2) = \left\{ p^* F_0(s_1, s_2) + \sum_i B_i \bar{F}_i(s_1, s_2) \right\} \quad (6.23)$$

It is readily seen that the above is a special case of F given by Equation (6.4). Using the last equation in the equilibrium equation (6.1) and by means of the Galerkin's method, the set of algebraic equations in the coefficients B_i 's can be shown to be

$$J_{op} + \sum_i \left\{ (J_1)_{ip} B_i + (J_2)_{ip} B_i^2 + \sum_j (J_4)_{ijp} (B_i B_j) \right\} = 0 \quad (6.24)$$

The above set consists of quadratic algebraic equations in the coefficients B_i 's. In fact, Equation (6.24) can be readily obtained from Equation (6.16) if the cubic terms (B_i^3 , $B_i^2 B_j$ and $B_i B_j B_k$ terms) are neglected. The set

of equations corresponding to linear buckling [that is, Equation (6.17)] can be obtained from Equation (6.24) if the nonlinear terms (B_1^2 and $B_1 u_j$ terms) are neglected. Therefore, Equation (6.24) which is based on the compatibility Equation (6.22) which is linear in W and F may be considered as the "first order postbuckling theory" of shells. Equation (6.24) which is valid for shells can be specialised for the case of flat plates. However, for flat plates, it must be remembered that the operator $\bar{D}(\rightarrow)$ vanishes so that Equation (6.22) reduces to $\nabla^4 F = 0$ which is the governing compatibility equation of linear buckling theory. Thus, for flat plates, there does not exist any "first order postbuckling theory"; we have only the linear buckling theory and the postbuckling theory based on Equations (6.1) and (6.2) wherein $\bar{D}(\rightarrow) = 0$.

If to the governing equations for the postbuckling problem of thin shells, the effects of damped vibration are included, Equation (6.1) has to include the terms $\rho \dot{W}$ and $C \dot{W}$ where ρ is the mass per unit area of the shell and C is the viscous damping coefficient. Thus, Equation (6.1) becomes:

$$\begin{aligned} \bar{L}(W, F) \equiv & \left[\rho \dot{W} + C \dot{W} + D(\nabla^4 W) - \left(\frac{N_{11}}{r_1} + \frac{N_{22}}{r_2} \right) \right. \\ & \left. - N_{11} \kappa_{11} - N_{22} \kappa_{22} - 2N_{12} \kappa_{12} - Z \right] = 0 \end{aligned} \quad (6.25)$$

The compatibility equation (6.2) remains unaltered. In the above, $W = W(s_1, s_2, t)$. Introducing the series for W given by Equation (6.3) where B_1 now is a generalised coordinate

and hence a function of time t , $\bar{L}(W, F)$ can be shown to be:

$$\bar{L}(W, F) = \sum_i (\rho \bar{W}_i) \ddot{B}_i + (c \bar{W}_i) \dot{B}_i + L(W, F) \quad (6.26)$$

where, $L(W, F)$ is given by Equation (6.13). Thus the modal equation obtained by using the Galerkin's method as in the last section becomes:

$$\begin{aligned} \sum_i \left\{ J_{ip} \left[\ddot{B}_i + \left(\frac{c}{\rho} \right) \dot{B}_i \right] + (J_1)_{ip} B_i + (J_2)_{ip} B_i^2 + (J_3)_{ip} B_i^3 \right. \\ \left. + \sum_j [(J_4)_{ijp} B_i B_j + (J_5)_{ijp} B_i^2 B_j] + \sum_j \sum_k (J_6)_{ijkp} B_i B_j B_k \right\} + J_{op} = 0 \end{aligned} \quad (6.27)$$

In the last equation, $J_{ip} = \rho \int_A \bar{W}_i \bar{W}_p dA$ and the other coefficients such as $(J_1)_{ip}$, $(J_2)_{ip}$ and so on are defined by Equation (6.15). If the set of functions $\{\bar{W}_i\}$ are orthogonal so that $\int_A \bar{W}_i \bar{W}_p dA = 0$ for $i \neq p$, the coefficients $J_{ip} = 0$ for $i \neq p$. It is important to remember that as one goes from the large amplitude vibration problem to the postbuckling problem, the signs of the stress resultants N_{11} , N_{22} and N_{12} have to be changed. Therefore, there will be a difference in the sign between $(\bar{F}_i)_{pb}$ and $(\bar{F}_i)_{fv}$ where the subscripts pb stand for the postbuckling problem and fv stand for the flexural vibration problem discussed in the

last section. Thus, $(\bar{F}_1)_{pb} = -(\bar{F}_1)_{fv}$. Similarly,

$$(\bar{F}_1)_{pb}'' = -(\bar{F}_1)_{fv}'' \text{ and } (F_1^* F_j^*)_{pt} = - (F_1^* F_j^*)_{fv}.$$

If the loading functions p^* and Z are zero, and the damping effect is neglected ($c = 0$), Equation (6.27) reduces to the modal equation developed in the last Section, namely, Equation (5.35). It should be recalled that when $p^* = 0$ and $Z = 0$, the coefficient $J_{op} = 0$. Also, if I_1 is independent of time t so that $\ddot{B}_1 = 0$ and $\dot{B}_1 = 0$, Equation (6.27) will reduce to Equation (6.16). Thus, Equation (6.27) is a very general equation from which the modal equation for the large amplitude flexural vibration of shells, plates and beams can be obtained. Also, the governing algebraic equations of the postbuckling problem of shells and plates as well as of the buckling problem (linear problem) of shells and plates can be obtained from Equation (6.27).

Conclusions.

The general equations governing the large amplitude flexural vibration of any thin elastic shell have been derived in this report using curvilinear orthogonal coordinates. They are two coupled nonlinear partial differential equations and they can be readily specialised for shells of revolution or flat plates. Although the material is assumed to be isotropic and to obey the Hooke's law for the two dimensional case, the geometric nonlinearity renders the governing equations nonlinear.

Based on a series expansion for the assumed mode shape, the modal equations are derived for the case of any thin flat plate and thin elastic shell. It is shown that the modal equation for flat plates is of a similar nature as the dynamic equation of a mass-spring combination in which the restoring force of the nonlinear spring is a cubic and an odd function of the displacement. Thus, the nonlinearity associated with the large amplitude flexural vibration of thin flat plates of any planform (rectangular, circular, elliptical and so on) is of the hardening type and such a conclusion is also true in the case of straight beams. On the other hand, the modal equation obtained for the case of any thin shell indicates that a softening type of nonlinearity is quite possible in the case of thin elastic shells. The modal equation for the shell can be readily specialised to yield the modal equation for a

shell of revolution or for a flat plate from which the modal equation for straight beams can then be obtained. The conclusions drawn above help to explain the reasons why in the existing literature, most of the available solutions for the large amplitude flexural vibration of thin shells are found to give rise to nonlinearity of the softening type whereas in the case of flat plates and straight beams, the nonlinearity is always found to be of the hardening type. The difference between the hardening type of nonlinearity in the case of flat plates and straight beams on the one hand and the softening type of nonlinearity in the case of rings and shells on the other hand can in general be traced to the curvature (or lack of it) of the undeformed median surface of the structure concerned.

When deriving the governing equations for the large amplitude flexural vibration of flat plates and thin shells, the governing equations for the postbuckling behavior of flat plates and thin shells can be obtained as a by-product. Study of these equations governing the postbuckling behavior of plates and shells leads to the following conclusions. In the case of the postbuckling behavior of flat plates, the algebraic equations in the coefficients of the assumed series for the postbuckling mode are a set of coupled, quadratic equations in these coefficients, whereas in the case of the postbuckling behavior of thin shells, the corresponding set of algebraic

equations are cubic; this situation is true when the lateral loading function $Z = 0$. If $Z \neq 0$, the set of algebraic equations in the case of both plates and shells consists of cubic equations in these coefficients. In general, the conclusions applicable for flat plates also hold true for straight beams and those applicable for shells hold good for rings.

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